QuestIQ Academy

JEE Advanced 2008 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Q1–23; Physics and Chemistry follow and are outside QuestIQ Academy's Math-only scope). Section 1 = single correct (Q1–6); Section 2 = one or more correct (Q7–10); Section 3 = assertion-reason (Q11–14); Section 4 = paragraph-based single correct (Q15–23).

Q.1

$C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$. *Relationship between the curves.*

Substituting $y^2 = 4x$ into C_2: $x^2 - 2x + 1 = (x-1)^2 = 0$, a repeated root at $x = 1$, giving points $(1, \pm 2)$. The double root signals tangency at each point – the curves touch at exactly two points.

Answer: (B) touch at exactly two points

Q.2

$0 < x < 1$: *simplify* $\sqrt{1+x^2}\left[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1\right]^{1/2}$.

With $\theta = \cot^{-1}x$: $\cos\theta = \frac{x}{\sqrt{1+x^2}}$, $\sin\theta = \frac{1}{\sqrt{1+x^2}}$, so $x\cos\theta + \sin\theta = \frac{x^2+1}{\sqrt{1+x^2}} = \sqrt{1+x^2}$. Then $\{\cdot\}^2 - 1 = x^2$, and the expression is $\sqrt{1+x^2}\cdot x = x\sqrt{1+x^2}$.

Answer: (C) $x\sqrt{1+x^2}$

Q.3

Parallelopiped edges = unit vectors $\hat{a}, \hat{b}, \hat{c}$ *with* $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = \frac{1}{2}$. *Find the volume.*

$$V^2 = \det\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix} = \frac{3}{4} - \frac{1}{8} - \frac{1}{8} = \frac{1}{2} \Rightarrow V = \frac{1}{\sqrt{2}}.$$

Answer: (A) $1/\sqrt{2}$

Q.4

$(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$, *$a, b$ nonzero. What does it represent?*

$x^2 - 5xy + 6y^2 = (x-2y)(x-3y)$ always gives two real lines. For $a = b$ and c opposite in sign to a: $ax^2 + ay^2 + c = 0 \Rightarrow x^2 + y^2 = -c/a > 0$, a genuine circle.

Answer: (B) two straight lines and a circle, when $a = b$ and c is of sign opposite to that of a

Q.5

$g(x) = \dfrac{(x-1)^n}{\log\cos^m(x-1)}$, $0 < x < 2$; $p =$ LHD of $|x-1|$ at $x = 1$ (so $p = -1$). If $\lim_{x\to 1^+} g(x) = p$, find n, m.

With $t = x - 1 \to 0^+$: $\log\cos^m t = m\log\cos t \approx -mt^2/2$, so $g \approx -\frac{2}{m}t^{n-2}$. Finiteness and matching $p = -1$ forces $n = 2$, then $-2/m = -1 \Rightarrow m = 2$.
Answer: (C) $n = 2, m = 2$

Q.6

$f(x) = (2+x)^3$ on $(-3,-1]$, $f(x) = x^{2/3}$ on $(-1, 2)$. Total local maxima + minima.

On $(-3,-1]$, $f' = 3(2+x)^2 \geq 0$ (non-decreasing, no interior extremum). On $(-1, 2)$, $f' = \frac{2}{3}x^{-1/3}$ changes sign at $x = 0$: local min there. At the junction $x = -1$ (where both pieces meet at value 1), f increases up to $x = -1$ then decreases just after: a local max at $x = -1$. Total: 2.
Answer: (C) 2

Q.7

Line through vertex P of $\triangle PQR$ meets QR at S and the circumcircle at T ($S \neq$ centre).

By the intersecting-chords (power of a point) theorem, $QS\cdot SR = PS\cdot ST$. AM–GM gives $\frac{1}{PS} + \frac{1}{ST} \geq \frac{2}{\sqrt{PS\cdot ST}}$, strict since S is not the midpoint of PT in general: $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS\times SR}}$. Since $QS\cdot SR \leq (QR/2)^2$, this also gives $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$.
Answer: (B), (D)

Q.8

$P, Q =$ endpoints ($y_1, y_2 < 0$) of the latus rectum of $x^2 + 4y^2 = 4$. Equations of parabolas with latus rectum PQ.

Here $a = 2, b = 1, c = \sqrt{3}$; the lower latus-rectum endpoints are $P = (\sqrt{3}, -\frac{1}{2}), Q = (-\sqrt{3}, -\frac{1}{2})$, a horizontal chord of length $2\sqrt{3}$ (so parabola parameter $a' = \sqrt{3}/2$). Two parabolas share this as latus rectum, opening up or down from focus $(0, -\frac{1}{2})$: $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$ and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$.
Answer: (B), (C)

Q.9

$S_n = \sum_{k=1}^{n} \dfrac{n}{n^2 + kn + k^2}$, $T_n = \sum_{k=0}^{n-1} \dfrac{n}{n^2 + kn + k^2}$.

Both are Riemann sums for $\int_0^1 \frac{dx}{1+x+x^2} = \frac{\pi}{3\sqrt{3}}$ (right sum S_n, left sum T_n), for the strictly decreasing integrand $\frac{1}{1+x+x^2}$. A right sum of a decreasing function underestimates the integral; a left sum overestimates it.

Answer: (A) $S_n < \pi/(3\sqrt{3})$**, (D)** $T_n > \pi/(3\sqrt{3})$

Q.10

Non-constant twice differentiable f with $f(x) = f(1-x)$, $f'(\frac{1}{4}) = 0$.

Differentiating gives $f'(x) = -f'(1-x)$; at $x = \frac{1}{2}$ this forces $f'(\frac{1}{2}) = 0$ (B), and by symmetry $f'(\frac{3}{4}) = 0$ too, giving zeros of f' at $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ – Rolle's theorem then forces f'' to vanish at least twice on $[0,1]$ (A). Substituting shows $f(\frac{1}{2} - x) = f(x + \frac{1}{2})$, so $f(x + \frac{1}{2}) \sin x$ is odd on $[-\frac{1}{2}, \frac{1}{2}]$, giving (C); and the substitution $s = 1 - t$ shows the two integrals in (D) are identical.
Answer: (A), (B), (C), (D)

Q.11

[Assertion-Reason] $g(0) \neq 0, g'(0) = 0, g''(0) \neq 0$ continuous, $f(x) = g(x) \sin x$.
S1: $\lim_{x\to 0}[g(x) \cot x - g(0) \csc x] = f'''(0)$. *S2:* $f'(0) = g(0)$.

$f'(0) = g(0) \sin\cdot\cos 0$-type direct computation gives $f'(0) = g(0)$: S2 is true. For S1, write the limit as $\lim \frac{g(x)\cos x - g(0)}{\sin x}$; L'Hopital once gives $\frac{g'(0)\cos 0 - g(0)\sin 0}{\cos 0} = g'(0) = 0$. But $f'''(0) = 3g''(0) - g(0)$ (via Leibniz, since the g''' term vanishes at $x = 0$), which is generically nonzero. A concrete check with $g(x) = 1 + x^2$ gives limit $= 0$ but $f'''(0) = 5$: S1 is false.
Answer: (D) STATEMENT-1 is False, STATEMENT-2 is True

Q.12

[Assertion-Reason] Planes $P_1 : x - y + z = 1$, $P_2 : x + y - z = -1$, $P_3 : x - 3y + 3z = 2$; $L_1 = P_2 \cap P_3$, $L_2 = P_3 \cap P_1$, $L_3 = P_1 \cap P_2$.
S1: at least two of L_1, L_2, L_3 non-parallel. S2: the three planes share no common point.

Direction vectors: $L_1 \parallel (0,1,1)$, $L_2 \parallel (0,1,1)$, $L_3 \parallel (0,1,1)$ – all three lines are mutually parallel, so S1 is false. Solving the system: adding P_1, P_2 gives $x = 0$, and substituting into P_3 yields $3 = 2$, a contradiction – no common point exists, so S2 is true.
Answer: (D) STATEMENT-1 is False, STATEMENT-2 is True

Q.13

[Assertion-Reason] $x - 2y + 3z = -1$, $-x + y - 2z = k$, $x - 3y + 4z = 1$.

S1: no solution for $k \neq 3$. *S2:* $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$ *for $k \neq 3$.*

The coefficient determinant is 0 (singular system). Direct elimination shows the system is consistent only at $k = 3$; for $k \neq 3$ it's inconsistent, confirming S1. The stated determinant evaluates to $3 - k$, vanishing exactly at $k = 3$ and nonzero otherwise – exactly the Cramer's-rule condition explaining S1.
Answer: (A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

Q.14

[Assertion-Reason] $ax + by = 0, cx + dy = 0,\ a, b, c, d \in \{0, 1\}$ uniformly random.
S1: $P(\text{unique solution}) = 3/8$. S2: $P(\text{has a solution}) = 1$.

Unique (trivial-only) solution needs $ad \neq bc$. Counting over all 16 equally likely (a, b, c, d): $ad = 1, bc = 0$ gives $1 \times 3 = 3$ cases; $ad = 0, bc = 1$ gives $3 \times 1 = 3$ cases; total $6/16 = 3/8$, confirming S1. S2 is trivially true (homogeneous systems always admit $x = y = 0$) but doesn't explain the specific probability in S1.

Answer: (B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

Q.15

[Paragraph] Incircle C (radius 1) of equilateral $\triangle PQR$; touches PQ at $D = (\frac{3\sqrt{3}}{2}, \frac{3}{2})$; line $PQ : \sqrt{3}x + y - 6 = 0$; origin and centre on the same side of PQ. Find circle C.

The origin gives $\sqrt{3}(0) + 0 - 6 < 0$, so the centre lies on the negative side, at distance 1 from D along the inward normal $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$: centre $= (\sqrt{3}, 1)$.

Answer: (D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Q.16

[Paragraph, same setup] Find points E, F (tangent points on QR, RP).

The radius-to-D vector is at $30°$; rotating by $\pm 120°$ locates E, F at $150°$ and $270°$ from the centre: $E = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $F = (\sqrt{3}, 0)$.

Answer: (A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$

Q.17

[Paragraph, same setup] Equations of sides QR, RP.

Tangent lines are perpendicular to their radii: at E (radius direction $150°$), the tangent has slope $\tan 60° = \sqrt{3}$, giving $QR : y = \sqrt{3}x$; at F (radius direction $270°$), tangent slope 0, giving $RP : y = 0$.

Answer: (D) $y = \sqrt{3}x,\ y = 0$

Q.18

[Paragraph] $y^3 - 3y + x = 0$ implicitly defines $y = f(x)$ on $(-\infty, -2) \cup (2, \infty)$ and $y = g(x)$ on $(-2, 2)$ with $g(0) = 0$. If $f(-10\sqrt{2}) = 2\sqrt{2}$, find $f''(-10\sqrt{2})$.

Implicit differentiation: $y' = \frac{1}{3(1-y^2)}$; at $y = 2\sqrt{2}$, $y' = -\frac{1}{21}$. Differentiating again: $y'' = \frac{-2y(y')^2}{y^2-1}$; with $y^2 - 1 = 7$: $y'' = \dfrac{-2(2\sqrt{2})(1/441)}{7} = -\dfrac{4\sqrt{2}}{7^3 \cdot 3^2}$.

Answer: (B) $-4\sqrt{2}/(7^3 3^2)$

Q.19

[Paragraph, same setup] Area bounded by $y = f(x)$, the x-axis, and $x = a, x = b$ $(-\infty < a < b < -2)$.

Since $f < 0$ on this range, area $= -\int_a^b f(x)\,dx$. Using the implicit relation $x = 3y - y^3$ (so $dx = 3(1-y^2)dy$) and integrating by parts in y converts this to the given closed form in terms of $f(a), f(b)$ and a residual integral.

Answer: (A) $\displaystyle\int_a^b \frac{x}{3\big((f(x))^2-1\big)}dx + bf(b) - af(a)$

Q.20

[Paragraph, same setup] Evaluate $\displaystyle\int_{-1}^{1} g'(x)\,dx$.

By the Fundamental Theorem of Calculus, this is $g(1) - g(-1)$. Since $g(0) = 0$ and g is odd about the origin on $(-2, 2)$ (as $y^3 - 3y + x = 0$ is odd in (x, y) jointly), $g(-1) = -g(1)$, so the integral is $g(1) - (-g(1)) = 2g(1)$.

Answer: (D) $2g(1)$

Q.21

[Paragraph] $A = \{z : \operatorname{Im} z \geq 1\}$, $B = \{z : |z - 2 - i| = 3\}$, $C = \{z : \operatorname{Re}((1-i)z) = \sqrt{2}\}$. Find $|A \cap B \cap C|$.

C is a line; B is a circle of radius 3 centred at $(2, 1)$. Solving the line equation with A's constraint and intersecting with the circle shows the line meets the circle in exactly one point satisfying $\operatorname{Im} z \geq 1$.

Answer: (B) 1

Q.22

[Paragraph, same sets] For $z \in A \cap B \cap C$, find the range of $|z + 1 - i|^2 + |z - 5 - i|^2$.

Using the parallelogram-law identity $|z-p|^2 + |z-q|^2 = 2|z-m|^2 + \frac{1}{2}|p-q|^2$ (with $m =$ midpoint of $p = -1+i, q = 5+i$, so $m = (2, 1)$ – the centre of circle B!) and $|z - m| = 3$ (radius) for the unique intersection point found above: the expression evaluates to a fixed value landing in this range.

Answer: (C) 35 and 39

Q.23

[Paragraph, same sets] For w satisfying $|w - 2 - i| \leq 3$, find the range of $|z| - |w| + 3$ (for z as in $A \cap B \cap C$).

With $|z|$ fixed by the unique point found above, and $|w|$ ranging over all points within distance 3 of $(2, 1)$ (i.e., $|w| \in [\sqrt{5} - 3, \sqrt{5} + 3]$), $|z| - |w| + 3$ ranges continuously across the stated bounds;

the official key lists this as accepting multiple option rows due to a flaw in the original problem's numerical data.

Answer: (B), (C), or (D) (multiple responses were accepted in the official key due to a data inconsistency in the original question)