QuestIQ Academy

JEE Advanced 2009 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = paragraph-based single correct; Section 4 = matrix-match.

Q.1

$P(3,2,6)$; Q on line $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(-3\hat{i} + \hat{j} + 5\hat{k})$. Find μ for which $\vec{PQ}$ is parallel to the plane $x - 4y + 3z = 1$.

$\vec{PQ} \cdot (1, -4, 3) = 0$ gives $-2 + 8\mu = 0 \Rightarrow \mu = 1/4$.

Answer: (A) $1/4$

Q.2

Normal at P on ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q; M = midpoint of PQ. Find where the locus of M meets the latus rectum.

With $P = (4\cos\theta, 2\sin\theta)$, the normal meets the axis at $Q = (3\cos\theta, 0)$, giving $M = (\frac{7}{2}\cos\theta, \sin\theta)$, i.e. $\frac{4X^2}{49} + Y^2 = 1$. At the latus rectum $x = \pm 2\sqrt{3}$: $Y^2 = 1 - \frac{48}{49} = \frac{1}{49}$.

Answer: (C) $(\pm 2\sqrt{3}, \pm 1/7)$

Q.3

Locus of the orthocentre of the triangle formed by $(1+p)x - py + p(1+p) = 0$, $(1+q)x - qy + q(1+q) = 0$, and $y = 0$ ($p \neq q$).

The vertices work out to $(-p, 0), (-q, 0), (pq, (p+1)(q+1))$. The altitude from the third vertex is vertical ($x = pq$); combined with the altitude from $(-p, 0)$, the orthocentre is $(pq, -pq)$ – tracing the line $y = -x$ as p, q vary.

Answer: (D) a straight line

Q.4

$z = x + iy$, x, y integers, satisfying $z\bar{z}^3 + \bar{z}z^3 = 350$. Area of the rectangle formed by the roots.

The equation reduces to $x^4 - y^4 = 175$, solved by $|x| = 4, |y| = 3$. The four roots $(\pm 4, \pm 3)$ form an axis-aligned rectangle of width 8 and height 6: area 48.

Answer: (A) 48

Q.5

Ellipse $x^2+9y^2=9$; line through extremities $A(3,0)$ of major axis and $B(0,1)$ of minor axis meets the auxiliary circle ($x^2+y^2=9$) at M. Find area of $\triangle AMO$.

Line AB: $x+3y=3$. Substituting into the auxiliary circle gives $M=(-\frac{12}{5},\frac{9}{5})$. Area $=\frac{1}{2}\left|3\cdot\frac{9}{5}\right|=\frac{27}{10}$.
Answer: (D) 27/10

Q.6

$\vec{a},\vec{b},\vec{c},\vec{d}$ *unit vectors,* $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=1$, $\vec{a}\cdot\vec{c}=\frac{1}{2}$.

Since each cross product has magnitude ≤ 1, equality to 1 forces $\vec{a}\perp\vec{b}$, $\vec{c}\perp\vec{d}$, both cross products equal (parallel, unit). Setting up coordinates in the shared plane, $\vec{b}\cdot\vec{d}=\cos(\alpha-\gamma)=\vec{a}\cdot\vec{c}=\frac{1}{2}\neq\pm1$, so $\vec{b},\vec{d}$ are not parallel (and $\vec{a},\vec{b},\vec{c}$ are in fact coplanar, ruling out (A),(B)).
Answer: (C) $\vec{b},\vec{d}$ **are non-parallel**

Q.7

$z=\cos\theta+i\sin\theta$. *Find* $\sum_{m=1}^{15}\mathrm{Im}(z^{2m-1})$ *at* $\theta=2^\circ$.

This is $\sum_{k=1}^{15}\sin((2k-1)\theta)=\dfrac{\sin^2(15\theta)}{\sin\theta}=\dfrac{\sin^2 30^\circ}{\sin 2^\circ}=\dfrac{1/4}{\sin 2^\circ}=\dfrac{1}{4\sin 2^\circ}$.
Answer: (D) $1/(4\sin 2^\circ)$

Q.8

Number of 7-digit integers, digit sum 10, using only digits 1, 2, 3.

With a,b,c copies of $1,2,3$: $a+b+c=7, a+2b+3c=10\Rightarrow b+2c=3$, giving $(a,b,c)=(4,3,0)$ or $(5,1,1)$. Arrangements: $\binom{7}{4,3,0}=35$ and $\binom{7}{5,1,1}=42$, total 77.
Answer: (C) 77

Q.9

Area of the region bounded by $y=e^x$, $x=0$, and $y=e$.

Direct integration gives area $=e-\int_0^1 e^x\,dx=e-(e-1)=1$; the alternative y-integrations $\int_1^e \ln y\,dy$ and $\int_1^e \ln(e+1-y)\,dy$ (via substitution) give the same value 1, while $e-1\approx 1.718\neq 1$.
Answer: (B), (C), (D)

Q.10

$L=\lim_{x\to0}\dfrac{a-\sqrt{a^2-x^2}-x^2/4}{x^4}$, $a>0$. *If L is finite, find a, L.*

Expanding $\sqrt{a^2 - x^2} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \cdots$, the numerator's x^2 coefficient is $\frac{1}{2a} - \frac{1}{4}$, which must vanish for finiteness: $a = 2$. Then $L = \frac{1}{8a^3} = \frac{1}{64}$.
Answer: (A) $a = 2$**, (C)** $L = 1/64$

Q.11

Triangle ABC with fixed base $BC = a$*; vertex* A *moves so that* $\cos B + \cos C = 4\sin^2(A/2)$*.*

Using $\cos B + \cos C = 2\sin(A/2)\cos(\frac{B-C}{2})$, the condition gives $\cos(\frac{B-C}{2}) = 2\sin(A/2)$. Combined with the sine rule, this leads to $b + c = 2a$; since a (the base) is fixed, $AB + AC =$ constant, so A traces an ellipse with foci B, C.
Answer: (B) $b + c = 2a$**, (C) locus of** A **is an ellipse**

Q.12

$\dfrac{\sin^4 x}{2} + \dfrac{\cos^4 x}{3} = \dfrac{1}{5}$. *Which hold?*

With $s = \sin^2 x$: $25s^2 - 20s + 4 = 0 = (5s - 2)^2$, giving $s = 2/5$ (a double root), so $\cos^2 x = 3/5$ and $\tan^2 x = 2/3$. Then $\sin^8 x/8 + \cos^8 x/27 = (2/5)^4/8 + (3/5)^4/27 = 2/625 + 3/625 = 1/125$.
Answer: (A), (B)

Q.13

[Paragraph] Let $\mathcal{A} = 3 \times 3$ *symmetric* $0/1$*-matrices with exactly five* 1*'s. Count* $|\mathcal{A}|$*.*

With D ones among the 3 diagonal entries and O ones among the 3 independent off-diagonal entries, $D + 2O = 5$: either $D = 3, O = 1$ ($1 \times 3 = 3$ matrices) or $D = 1, O = 2$ ($3 \times 3 = 9$ matrices), total 12.
Answer: (A) 12

Q.14

[Paragraph, same $\mathcal{A}$*] Count* $A \in \mathcal{A}$ *for which* $Ax = (1, 0, 0)^T$ *has a unique solution.*

Checking determinants case by case: all 3 matrices with $D = 3, O = 1$ are singular; among the 9 with $D = 1, O = 2$, exactly 6 are non-singular (by direct computation and symmetry). Total invertible: 6.
Answer: (B) at least 4 but less than 7

Q.15

[Paragraph, same $\mathcal{A}$*] Count* $A \in \mathcal{A}$ *for which the system is inconsistent.*

Of the 6 singular matrices, direct row-reduction shows 2 (from the $D = 3, O = 1$ family) give a genuine contradiction (inconsistent), while the remaining 4 admit infinitely many solutions.
Answer: (C) 2

Q.16

[Paragraph] Die tossed until a six appears; X = number of tosses. Find $P(X = 3)$.

$P(X = 3) = \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} = \frac{25}{216}$.

Answer: (A) 25/216

Q.17

[Paragraph, same setup] Find $P(X \geq 3)$.

$P(X \geq 3) = 1 - P(X = 1) - P(X = 2) = 1 - \frac{1}{6} - \frac{5}{36} = \frac{25}{36}$.

Answer: (B) 25/36

Q.18

[Paragraph, same setup] Find $P(X \geq 6 \mid X > 3)$.

By the memoryless property, conditioning on $X > 3$ restarts the geometric process; $P(X \geq 6 \mid X > 3) = P(Y \geq 3) = (5/6)^2 = 25/36$ for the restarted count Y.

Answer: (D) 25/36

Q.19

[Matrix match] Match Column I intervals with Column II ($p = (-\frac{\pi}{2}, \frac{\pi}{2})$, $q = (0, \frac{\pi}{2})$, $r = (\frac{\pi}{8}, \frac{5\pi}{4})$, $s = (0, \frac{\pi}{8})$, $t = (-\pi, \pi)$).

(A) The domain of non-zero solutions of $(x-3)^2 y' + y = 0$ excludes $x = 3$: only r, t contain 3, so (A)$\to p, q, s$. (B) $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx = 0$ (odd about $x = 3$ after shifting); only p, t contain 0. (C) $\cos^2 x + \sin x$ has local maxima at $x = \pi/6 + 2k\pi, 5\pi/6 + 2k\pi$; checking containment, p, q, r, t each contain $\pi/6$ or $5\pi/6$. (D) $\tan^{-1}(\sin x + \cos x)$ increases exactly where $\cos x > \sin x$, i.e. on $(-\frac{3\pi}{4}, \frac{\pi}{4})$; only $s = (0, \pi/8)$ is fully contained in this range.

Answer: A→p,q,s; B→p,t; C→p,q,r,t; D→s

Q.20

[Matrix match] Match conics to their descriptions (p: locus of (h, k) where $hx + ky = 1$ touches $x^2 + y^2 = 4$; q: $|z+2| - |z-2| = \pm 3$; r: parametric $x = \sqrt{3}\frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$; s: eccentricity in $[1, \infty)$; t: $\mathrm{Re}(z+1)^2 = |z|^2 + 1$).

(A) Circle: p gives $h^2 + k^2 = 1/4$, a circle – matches. (B) Parabola: t simplifies to $x = y^2$, a parabola, and its eccentricity $= 1 \in s$. (C) Ellipse: r traces $\frac{x^2}{3} + y^2 = 1$, an ellipse. (D) Hyperbola: q is the standard $|PF_1| - |PF_2|$ =const definition of a hyperbola, with eccentricity $> 1 \in s$.

Answer: A→p; B→s,t; C→r; D→q,s