QuestIQ Academy

JEE Advanced 2009 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = matrix-match; Section 4 = single-digit integer.

Q.1

Line through $P(2,-1,2)$ with positive, equal direction cosines meets the plane $2x+y+z=9$ at Q. Find PQ.

Direction $= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Parametrising and substituting into the plane: $5+\frac{4t}{\sqrt{3}} = 9 \Rightarrow t = \sqrt{3}$. Since t is the arc-length parameter, $PQ = \sqrt{3}$.
Answer: (C) $\sqrt{3}$

Q.2

Normal at P on ellipse $x^2+4y^2 = 16$ meets the x-axis at Q; M = midpoint of PQ. Find where the locus of M meets the latus rectum.

As in Paper 1: M traces $\frac{4X^2}{49} + Y^2 = 1$, meeting $x = \pm 2\sqrt{3}$ at $Y = \pm 1/7$.
Answer: (C) $(\pm 2\sqrt{3}, \pm 1/7)$

Q.3

Locus of the orthocentre of the triangle formed by $(1+p)x-py+p(1+p) = 0$, $(1+q)x-qy+q(1+q) = 0$, and $y = 0$ ($p \neq q$).

As in Paper 1: orthocentre $= (pq, -pq)$, tracing the line $y = -x$.
Answer: (D) a straight line

Q.4

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}\, dx,\ n = 0, 1, 2, \ldots$$

Substituting $x \to -x$ and adding to the original shows $2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$, and this classical integral equals 2π for odd n, 0 for even n. So $I_n = \pi$ (n odd), $I_n = 0$ (n even).
Answer: (A) $I_n = I_{n+2}$, (B) $\sum_{m=1}^{10} I_{2m+1} = 10\pi$, (C) $\sum_{m=1}^{10} I_{2m} = 0$

Q.5

Ellipse (axes on coordinate axes) intersects $2x^2 - 2y^2 = 1$ orthogonally; ellipse eccentricity is the reciprocal of the hyperbola's.

Hyperbola: $a_h^2 = b_h^2 = \frac{1}{2}$, $e_h = \sqrt{2}$, foci at $(\pm 1, 0)$. So ellipse eccentricity $e = 1/\sqrt{2}$. Orthogonally-intersecting confocal conics share foci, so the ellipse also has foci $(\pm 1, 0)$: $Ae = 1 \Rightarrow A^2 = 2$, $B^2 = A^2(1 - e^2) = 1$, giving $x^2 + 2y^2 = 2$.

Answer: (A) equation of ellipse is $x^2 + 2y^2 = 2$, (B) foci at $(\pm 1, 0)$

Q.6

$f(x) = x\cos(1/x)$, $x \geq 1$.

$f'(x) = \cos(1/x) + \frac{1}{x}\sin(1/x)$. As $x \to \infty$: $f'(x) \to 1$. Writing $u = 1/x$, $f' = \cos u + u \sin u \approx 1 + \frac{u^2}{2} > 1$ for all finite x, so $f(x+2) - f(x) = 2f'(c) > 2$ for all $x \geq 1$ (mean value theorem). Also $\dfrac{d}{du}(\cos u + u\sin u) = u\cos u > 0$ for $u \in (0, 1)$, so f' (as a function of decreasing $u = 1/x$) is strictly decreasing in x.

Answer: (B) $\lim_{x\to\infty} f'(x) = 1$, (C) $f(x+2) - f(x) > 2$ for all $x \geq 1$, (D) f' is strictly decreasing on $[1, \infty)$

Q.7

Tangent and normal to $y^2 = 4ax$ at $P(at^2, 2at)$ meet the axis at T, N. Find properties of the locus of the centroid of $\triangle PTN$.

$T = (-at^2, 0)$, $N = (at^2 + 2a, 0)$. Centroid $G = \left(\frac{at^2+2a}{3}, \frac{2at}{3}\right)$, giving $Y^2 = \frac{4a}{3}\left(X - \frac{2a}{3}\right)$: a parabola with vertex $\left(\frac{2a}{3}, 0\right)$ and focus $\left(\frac{2a}{3} + \frac{a}{3}, 0\right) = (a, 0)$.

Answer: (A) vertex is $\left(\frac{2a}{3}, 0\right)$, (D) focus is $(a, 0)$

Q.8

$0 < \theta < \pi/2$: solve $\displaystyle\sum_{m=1}^{6} \csc\left(\theta + \frac{(m-1)\pi}{4}\right)\csc\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$.

Using $\csc A \csc B = \sqrt{2}(\cot A - \cot B)$ for $B - A = \pi/4$, the sum telescopes to $\sqrt{2}\left[\cot\theta - \cot\left(\theta + \frac{3\pi}{2}\right)\right] = \sqrt{2}(\cot\theta + \tan\theta) = \dfrac{2\sqrt{2}}{\sin 2\theta}$. Setting this to $4\sqrt{2}$ gives $\sin 2\theta = \frac{1}{2}$, so $2\theta = \pi/6$ or $5\pi/6$.

Answer: (C) $\pi/12$, (D) $5\pi/12$

Q.9

[Matrix match] Match Column I to Column II ($p = \pi/6, q = \pi/4, r = \pi/3, s = \pi/2, t = \pi$).

(A) $2\sin^2\theta + \sin^2 2\theta = 2$ reduces to $2\sin^2\theta - 1)(\sin^2\theta - 1) = 0$-type factoring, giving $\theta = \pi/4$ or $\pi/2$ among the roots. (B) points of discontinuity of $f(x) = \lfloor 6x/\pi \rfloor \cos(3x/\pi)$ occur at $x = \pi/6$ and $x = \pi/2$ (where the floor jumps and cosine doesn't vanish). (C) volume of the parallelepiped

with edges $\hat{i}+\hat{j}$, $\hat{i}+2\hat{j}$, $\hat{i}+\hat{j}+\pi\hat{k}$ is π. (D) from $\vec{a}+\vec{b}+\sqrt{3}\vec{c}=\vec{0}$ (unit vectors): $\vec{a}\cdot\vec{b}=\frac{1}{2}$, so the angle between $\vec{a},\vec{b}$ is $\pi/3$.
Answer: A→q,s; B→p,s; C→t; D→r

Q.10

[Matrix match] Match Column I to Column II ($p=1,q=2,r=3,s=4,t=5$).

(A) $g(x)=xe^{\sin x}-\cos x$ satisfies $g(0)=-1<0$, $g(\pi/2)>0$, and $g'(x)>0$ throughout $(0,\pi/2)$: exactly one root. (B) planes $kx+4y+z=0, 4x+ky+2z=0, 2x+2y+z=0$ meet in a line when $\det=0$: $k^2-6k+8=0 \Rightarrow k=2$ or 4. (C) $|x-1|+|x-2|+|x+1|+|x+2|=4k$ has an integer solution for $k=2,3,4,5$ (sums $8,12,16,20$ at $x=\pm2,3,4,5$), but not $k=1$ (minimum achievable sum is 6). (D) $y'=y+1, y(0)=1 \Rightarrow y=2e^x-1$; at $x=\ln 2$, $y=3$.
Answer: A→p; B→q,s; C→q,r,s,t; D→r

Q.11

Max of $f(x)=2x^3-15x^2+36x-48$ on $A=\{x : x^2+20\le 9x\}$.

$A=[4,5]$. $f'(x)=6(x-2)(x-3)>0$ on $[4,5]$, so f is increasing there; max at $x=5$: $f(5)=250-375+180-48=7$.
Answer: 7

Q.12

Integer solutions (x,y,z) of $3x-y-z=0, -3x+z=0, -3x+2y+z=0$ with $x^2+y^2+z^2\le 100$. Find the count.

Solving gives $z=3x, y=0$; the constraint becomes $10x^2\le 100 \Rightarrow |x|\le\sqrt{10}$, so $x\in\{-3,\dots,3\}$: 7 values.
Answer: 7

Q.13

Non-congruent $\triangle ABC, ABC'$: $AB=4, AC=AC'=2\sqrt{2}, \angle B=30^\circ$. Find $|Area(ABC)-Area(ABC')|$.

Law of sines gives $\sin C=1/\sqrt{2}$, so $C=45^\circ$ or 135°, with $A=105^\circ$ or 15° respectively. Areas are $4\sqrt{2}\sin 105^\circ$ and $4\sqrt{2}\sin 15^\circ$; difference $=4\sqrt{2}(\sin 105^\circ-\sin 15^\circ)=4\sqrt{2}\cdot\frac{1}{\sqrt{2}}=4$.
Answer: 4

Q.14

Degree-4 polynomial p with extrema at $x=1,2$; $\lim_{x\to 0}\left[1+\frac{p(x)}{x^2}\right]=2$. Find $p(2)$.

Finiteness forces $p(x)=ax^4+bx^3+x^2$. From $p'(1)=p'(2)=0$: $a=1/4, b=-1$. Then $p(2)=4-8+4=0$.
Answer: 0

Q.15

$f : \mathbb{R} \to \mathbb{R}$ continuous with $f(x) = \int_0^x f(t)\,dt$ for all x. Find $f(\ln 5)$.

Differentiating gives $f' = f$ with $f(0) = 0$, forcing $f \equiv 0$.

Answer: 0

Q.16

Two unit circles C_1, C_2 with centres 6 apart, P = midpoint. Circle C touches both externally; the common tangent to C_1, C through P is also a common tangent to C_2, C. Find the radius of C.

Standard result for this classic configuration: radius = 8.

Answer: 8

Q.17

Smallest k for which both roots of $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct, and ≥ 4.

Discriminant $> 0 \Rightarrow k > 1$. Vertex condition $4k \geq 4 \Rightarrow k \geq 1$. $f(4) \geq 0 \Rightarrow (k-1)(k-2) \geq 0 \Rightarrow k \leq 1$ or $k \geq 2$. Combined: $k \geq 2$.

Answer: 2

Q.18

$f(x) = x^3 + e^{x/2}$, $g = f^{-1}$. Find $g'(1)$.

$f(0) = 1 \Rightarrow g(1) = 0$. $f'(0) = \frac{1}{2}$. $g'(1) = 1/f'(0) = 2$.

Answer: 2