QuestIQ Academy

JEE Advanced 2011 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = single-digit integer; Section 4 = matrix-match.

Q.1

$P(6,3)$ on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; normal at P meets the x-axis at $(9,0)$. Find the eccentricity.

The normal meets the x-axis at $x = x_1(a^2+b^2)/a^2$; setting this to 9 with $x_1 = 6$ gives $a^2 = 2b^2$. Using P on the curve, $36/a^2 - 9/b^2 = 1$ with $b^2 = a^2/2$ gives $a^2 = 18, b^2 = 9$, so $e^2 = 1 + b^2/a^2 = 3/2$.
Answer: (B) $\sqrt{3/2}$

Q.2

$x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have one common root. Find b.

Eliminating the common root gives $r = (1+b)/(b-1)$; substituting back into either equation yields $b^3 + 3b = 0$, so $b = 0$ or $b = \pm i\sqrt{3}$.
Answer: (B) $-i\sqrt{3}$

Q.3

$\omega \neq 1$ a cube root of unity; S = non-singular matrices $\begin{pmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{pmatrix}$, $a, b, c \in \{\omega, \omega^2\}$. Number of distinct matrices in S.

The determinant simplifies to $1 - \omega(a+c) + ac\,\omega^2$, which is independent of b. Checking all four (a,c) combinations, only $a = c = \omega$ gives a nonzero determinant ($-3\omega^2 \neq 0$); this works for both choices of b, giving 2 non-singular matrices.
Answer: (A) 2

Q.4

Circle through $(-1,0)$, touching the y-axis at $(0,2)$. Which other point lies on it?

Centre is $(h, 2)$ with radius $|h|$: $(x-h)^2 + (y-2)^2 = h^2$. Through $(-1, 0)$: $(1+h)^2 + 4 = h^2 \Rightarrow h = -\frac{5}{2}$. Checking each option against $(x + \frac{5}{2})^2 + (y-2)^2 = \frac{25}{4}$, only $(-4, 0)$ satisfies it.
Answer: (D) $(-4, 0)$

Q.5

$\lim_{x\to 0} \left[1 + x\ln(1+b^2)\right]^{1/x} = 2b\sin^2\theta$, *$b > 0$, $\theta \in (-\pi, \pi]$. Find θ.*

The limit equals $1 + b^2$. By AM–GM, $1 + b^2 \geq 2b$, so $2b\sin^2\theta \geq 2b \Rightarrow \sin^2\theta \geq 1 \Rightarrow \sin^2\theta = 1$, forcing $\theta = \pm\pi/2$ (and consequently $b = 1$).
Answer: (D) $\pm\pi/2$

Q.6

$f : [-1, 2] \to [0, \infty)$ *continuous with* $f(x) = f(1-x)$. $R_1 = \int_{-1}^{2} xf(x)dx$, $R_2 = \int_{-1}^{2} f(x)dx$. *Relation between R_1, R_2?*

Substituting $x \to 1-x$ in R_1 and using $f(1-x) = f(x)$ gives $R_1 = \int_{-1}^{2}(1-x)f(x)dx = R_2 - R_1$, so $2R_1 = R_2$.
Answer: (C) $2R_1 = R_2$

Q.7

$f(x) = x^2, g(x) = \sin x$. *Solve* $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$.

Let $w = \sin(\sin x^2)$; the equation becomes $w^2 = w$. Since $|\sin(\text{anything in } [-1, 1])| < 1$, $w = 1$ is impossible, so $w = 0$, giving $\sin(x^2) = 0$ (as $\sin x^2 \in [-1, 1]$, only the $n = 0$ multiple of π is reachable... more precisely $x^2 = n\pi$), i.e. $x = \pm\sqrt{n\pi}$, $n = 0, 1, 2, \ldots$
Answer: (A) $\pm\sqrt{n\pi}$, $n \in \{0, 1, 2, \ldots\}$

Q.8

(x, y) *on* $y^2 = 4x$; *P divides $(0, 0)$ to (x, y) in ratio $1 : 3$. Locus of P.*

$P = (x/4, y/4)$. Writing $X = x/4, Y = y/4$: $x = 4X, y = 4Y$, and $y^2 = 4x \Rightarrow 16Y^2 = 16X \Rightarrow Y^2 = X$.
Answer: (C) $y^2 = x$

Q.9

Piecewise $f(x)$: $-x - \frac{\pi}{2}$ $(x \leq -\frac{\pi}{2})$, $-\cos x$ $(-\frac{\pi}{2} < x \leq 0)$, $x - 1$ $(0 < x \leq 1)$, $\ln x$ $(x > 1)$. Which hold?

At $x = -\frac{\pi}{2}$: both pieces give 0 (continuous) – (A). At $x = 0$: left derivative $\sin(0) = 0 \neq 1 =$ right derivative, so not differentiable – (B). At $x = 1$: both derivatives equal 1, so differentiable – (C). At $x = -\frac{3}{2}$ (an interior point of the smooth $-\cos x$ piece), f is trivially differentiable – (D).
Answer: (A), (B), (C), (D)

Q.10

E, F independent; $P(\text{exactly one}) = \frac{11}{25}$, $P(\text{none}) = \frac{2}{25}$. Find $P(E), P(F)$.

$P(E) + P(F) - P(E)P(F) = \frac{23}{25}$ and (exactly one)$= \frac{23}{25} - P(E)P(F) = \frac{11}{25} \Rightarrow P(E)P(F) = \frac{12}{25}$, giving $P(E) + P(F) = \frac{7}{5}$. Solving $t^2 - \frac{7}{5}t + \frac{12}{25} = 0$ gives $\{P(E), P(F)\} = \{\frac{3}{5}, \frac{4}{5}\}$.

Answer: (A), (D)

Q.11

L normal to $y^2 = 4x$, passes through $(9, 6)$. Find L.

Normal at parameter t: $y + tx = 2t + t^3$. Through $(9, 6)$: $t^3 - 7t - 6 = 0 = (t + 1)(t - 3)(t + 2)$, giving $t = -1, 3, -2$ and lines $y - x + 3 = 0$, $y + 3x - 33 = 0$, $y - 2x + 12 = 0$.

Answer: (A), (B), (D)

Q.12

$f : (0, 1) \to \mathbb{R}$, $f(x) = \dfrac{b - x}{1 - bx}$, $0 < b < 1$.

Direct computation shows $f(f(x)) = x$ (an involution) and f is injective. However $f(1^-) \to -1 \notin (0, 1)$, so f does not map $(0, 1)$ into itself – hence f is not invertible *as a self-map of* $(0, 1)$, even though it is injective onto its actual range.

Answer: (A) f is not invertible on $(0, 1)$

Q.13

$\omega = e^{i\pi/3}$; a, b, c non-zero complex with $a + b + c = x$, $a + b\omega + c\omega^2 = y$, $a + b\omega^2 + c\omega = z$. Find $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$.

This question was officially voided (marks to all candidates) – likely because ω here is a primitive sixth root of unity rather than the cube root normally used in this three-term transform identity, making the intended closed-form ratio (usually 3 for a genuine DFT-type setup) inapplicable as stated.

Answer: Officially voided (marks to all)

Q.14

Number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$.

$f''(x) = 12(x - 1)^2 + 12 > 0$ always, so $f'(x)$ is strictly increasing and has exactly one real root $r \approx -0.04$, giving f a single global minimum. Evaluating $f(r) \approx -1.02 < 0$, and since $f \to +\infty$ at both ends, f crosses zero exactly twice.

Answer: 2

Q.15

$y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, g non-constant differentiable with $g(0) = g(2) = 0$. Find $y(2)$.

Integrating factor $e^{g(x)}$ gives $(ye^{g(x)})' = g(x)g'(x)e^{g(x)}$. Integrating from 0 to 2: LHS $= y(2)e^{g(2)} - y(0)e^{g(0)} = y(2)$; RHS, substituting $u = g(x)$, becomes $\int_{g(0)}^{g(2)} ue^u\,du = \int_0^0 ue^u\,du = 0$.
Answer: 0

Q.16

3×3 matrix M satisfies $M(0,1,0)^T = (-1,2,3)^T$, $M(1,-1,0)^T = (1,1,-1)^T$, $M(1,1,1)^T = (0,0,12)^T$. Find $\operatorname{tr}(M)$.

With V the matrix of input columns and W the matrix of output columns, $M = WV^{-1}$. Computing V^{-1} (using $\det V = -1$) and the trace via $\operatorname{tr}(M) = \sum_{i,k} W_{ik}(V^{-1})_{ki}$ gives contributions $0 + 2 + 7 = 9$.
Answer: 9

Q.17

$\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$, $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$; $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$, $\vec{r} \cdot \vec{a} = 0$. Find $\vec{r} \cdot \vec{b}$.

$(\vec{r} - \vec{c}) \times \vec{b} = 0 \Rightarrow \vec{r} = \vec{c} + \lambda\vec{b} = (1-\lambda, 2+\lambda, 3)$. Imposing $\vec{r} \cdot \vec{a} = 0$ gives $\lambda = 4$, so $\vec{r} = (-3, 6, 3)$ and $\vec{r} \cdot \vec{b} = 3 + 6 + 0 = 9$.
Answer: 9

Q.18

Line $2x - 3y = 1$ divides the disk $x^2 + y^2 \le 6$ into two parts. How many of the four given points lie in the smaller part?

The centre satisfies $2(0) - 3(0) = 0 < 1$, so the smaller (minor) segment is where $2x - 3y > 1$. Checking each point: $(2, \frac{3}{4})$ gives $1.75 > 1$ and lies inside the disk (counts); $(\frac{5}{2}, \frac{3}{4})$ lies *outside* the disk entirely (excluded); $(\frac{1}{4}, -\frac{1}{4})$ gives $1.25 > 1$ and is inside the disk (counts); $(\frac{1}{8}, \frac{1}{4})$ gives $-0.5 < 1$ (in the larger part, excluded).
Answer: 2

Q.19

[Matrix match] Match Column I to Column II.

(A) With $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}, \vec{c} = 2\sqrt{3}\hat{k}$ as triangle sides ($|\vec{a}| = |\vec{b}| = 2, |\vec{c}| = 2\sqrt{3}$), the law of cosines gives the angle between $\vec{a}, \vec{b}$ as $2\pi/3$. (B) Differentiating $\int_a^b (f(x) - 3x)dx = a^2 - b^2$ with respect to b gives $f(x) = x$, so $f(\pi/6) = \pi/6$. (C) $\frac{\pi^2}{\ln 3}\int_{1/6}^{5/6} \sec(\pi x)dx$ evaluates (via the standard antiderivative, officially) to π. (D) $\max|\operatorname{Arg}(1/(1-z))|$ for $|z| = 1, z \ne 1$ approaches $\pi/2$ as $z \to 1$.
Answer: A→q ($2\pi/3$), B→p ($\pi/6$; official key also accepts p,q,r,s,t), C→s (π), D→t ($\pi/2$)

Q.20

[Matrix match] Match Column I to Column II.

(A) With $z = e^{i\theta}$, $\mathrm{Re}\left(\frac{2iz}{1-z^2}\right) = -\frac{1}{\sin\theta}$, which (as θ ranges over all valid values, $z \neq \pm 1$) covers exactly $(-\infty, -1] \cup [1, \infty)$. (B) Substituting $t = 3^{x-1}$, the domain condition reduces to $t \leq \frac{1}{3}$ or $t \geq 3$, giving $x \leq 0$ or $x \geq 2$. (C) The determinant simplifies to $2\sec^2\theta$, ranging over $[2, \infty)$ for $\theta \in [0, \pi/2)$. (D) $f'(x) = \frac{15}{2}\sqrt{x}(x-2) \geq 0$ for $x \geq 2$, so f increases on $[2, \infty)$.

Answer: A→s, B→t, C→r, D→r