QuestIQ Academy

JEE Advanced 2012 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = single-digit integer.

Q.1

5 balls of different colours distributed among 3 persons so each gets at least one.

This is the number of onto functions from a 5-element set to a 3-element set: $3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5 = 243 - 96 + 3 = 150$.

Answer: (B) 150

Q.2

$f(x) = x^2|\cos(\pi/x)|, x \neq 0$; $f(0) = 0$. *Differentiability at* $x = 0$ *and* $x = 2$*?*

At $x = 0$: $|f(h)/h| = h|\cos(\pi/h)| \to 0$, so $f'(0) = 0$ exists. At $x = 2$: $g(x) = \cos(\pi/x)$ has $g(2) = \cos(\pi/2) = 0$ with $g'(2) = \sin(\pi/2) \cdot \pi/4 \neq 0$, so $|g(x)|$ has a corner at $x = 2$; since $x^2 = 4 \neq 0$ there, $f = x^2|g(x)|$ inherits the non-differentiability.

Answer: (B) differentiable at $x = 0$ but not $x = 2$

Q.3

$f : [0, 3] \to [1, 29]$, $f(x) = 2x^3 - 15x^2 + 36x + 1$. *Classify* f.

$f'(x) = 6(x - 2)(x - 3)$: f increases on $[0, 2]$ from 1 to 29, then decreases on $[2, 3]$ to 28. The range $[1, 29]$ is fully covered by the increasing branch alone (onto), but $f(3) = 28$ repeats a value already attained while increasing, so f is not one-one.

Answer: (B) onto but not one-one

Q.4

$\lim_{x\to\infty} \left(\dfrac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$. *Find* a, b.

$\dfrac{x^2 + x + 1}{x + 1} = x + \dfrac{1}{x + 1}$, so the expression is $(1 - a)x - b + \dfrac{1}{x + 1}$. Finiteness forces $a = 1$; then the limit is $-b = 4 \Rightarrow b = -4$.

Answer: (B) $a = 1, b = -4$

Q.5

$z = x + iy$, $y \neq 0$, $a = z^2 + z + 1$ real. Which value can a not take?

The imaginary part is $y(2x+1) = 0 \Rightarrow x = -\frac{1}{2}$ (since $y \neq 0$). Then $a = x^2 - y^2 + x + 1 = \frac{3}{4} - y^2 < \frac{3}{4}$ for all $y \neq 0$. So a can never equal (or exceed) $\frac{3}{4}$.
Answer: (D) 3/4

Q.6

Ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ inscribed in rectangle R; E_2 through $(0,4)$ circumscribes R. Eccentricity of E_2.

R has corners $(\pm 3, \pm 2)$. For $E_2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ through $(0,4)$: $B = 4$. Through $(3,2)$: $\frac{9}{A^2} + \frac{4}{16} = 1 \Rightarrow A^2 = 12$. Eccentricity $= \sqrt{1 - A^2/B^2} = \sqrt{1 - 12/16} = \frac{1}{2}$.
Answer: (C) 1/2

Q.7

$P = [a_{ij}]$ 3×3, $Q = [b_{ij}]$, $b_{ij} = 2^{i+j} a_{ij}$, $\det P = 2$. Find $\det Q$.

$Q = D_1 P D_2$ with $D_1 = D_2 = \mathrm{diag}(2,4,8)$, so $\det Q = (2 \cdot 4 \cdot 8)^2 \det P = 64^2 \cdot 2 = 2^{13}$.
Answer: (D) 2^{13}

Q.8

$$\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx.$$

Differentiating option (C), $F = -(\sec x + \tan x)^{-11/2} \left[\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right]$, and using $u = \sec x + \tan x$, $u' = u \sec x$, $u^2 + 1 = 2u \sec x$, gives $F'(x) = \sec^2 x \, u^{-9/2}$, exactly the integrand.
Answer: (C)

Q.9

P = intersection of line QR ($Q(2,3,5), R(1,-1,4)$) with plane $5x - 4y - z = 1$. S = foot of perpendicular from $T(2,1,4)$ to QR. Find PS.

Line QR: $(2 - t, 3 - 4t, 5 - t)$; substituting into the plane gives $t = \frac{2}{3}$, so $P = (\frac{4}{3}, \frac{1}{3}, \frac{13}{3})$. Projecting $T - Q = (0, -2, -1)$ onto direction $(-1, -4, -1)$ (magnitude$^2 = 18$) gives parameter $\frac{1}{2}$, so $S = (\frac{3}{2}, 1, \frac{9}{2})$. Then $PS = \sqrt{(-\frac{1}{6})^2 + (-\frac{4}{6})^2 + (-\frac{1}{6})^2} = \sqrt{1/2} = 1/\sqrt{2}$.
Answer: (A) $1/\sqrt{2}$

Q.10

Locus of midpoint of the chord of contact from points on $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$.

For external point (h, k), the chord of contact $hx+ky = 9$ has midpoint $(X, Y) = \left(\frac{9h}{h^2+k^2}, \frac{9k}{h^2+k^2}\right)$, so $h = \frac{9X}{X^2+Y^2}, k = \frac{9Y}{X^2+Y^2}$. Substituting into $4h - 5k = 20$ gives $20(X^2 + Y^2) - 36X + 45Y = 0$.
Answer: (A) $20(x^2 + y^2) - 36x + 45y = 0$

Q.11

$\theta, \varphi \in [0, 2\pi]$: $2\cos\theta(1-\sin\varphi) = \sin^2\theta(\tan\frac{\theta}{2}+\cot\frac{\theta}{2})\cos\varphi-1$, $\tan(2\pi-\theta) > 0$, $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$. Which range can φ not satisfy?

Since $\tan\frac{\theta}{2} + \cot\frac{\theta}{2} = \frac{2}{\sin\theta}$, the equation simplifies to $\sin(\theta+\varphi) = \cos\theta + \frac{1}{2}$. The constraints force $\theta \in (\frac{3\pi}{2}, \frac{5\pi}{3})$, so $\cos\theta \in (0, \frac{1}{2})$. Tracing both inverse-sine branches as θ ranges over this interval shows the achievable φ values lie strictly inside $(\frac{2\pi}{3}, \frac{4\pi}{3}) \subset \left(\frac{\pi}{2}, \frac{4\pi}{3}\right)$ – entirely within option (B)'s range, so φ can never fall in the other three ranges.
Answer: (A), (C), (D)

Q.12

$S = \displaystyle\int_0^1 e^{-x^2}dx$. *Which bound(s) hold?*

Using $e^{-x^2} \geq 1 - x^2$ (from $e^t \geq 1 + t$), $S \geq \int_0^1(1 - x^2)dx = \frac{2}{3}$, which already exceeds both $\frac{1}{e}$ and $1 - \frac{1}{e}$ (options A, B). Numerically $S \approx 0.747$, which is well below option (D)'s bound ≈ 0.885, but exceeds option (C)'s bound ≈ 0.402 (so (C) is false).
Answer: (A), (B), (D)

Q.13

Ship with engines E_1, E_2, E_3, probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$; operational ($X$) needs ≥ 2 functioning. Which is (are) true?

$P(X) = P(\text{all 3}) + P(\text{exactly 2}) = \frac{1}{32} + \frac{7}{32} = \frac{1}{4}$. Then $P(\text{exactly two} \mid X) = \frac{7/32}{1/4} = \frac{7}{8}$ (B), and $P(X \mid X_1) = P(\text{at least one of } E_2, E_3) = 1 - \frac{3}{4}\cdot\frac{3}{4} = \frac{7}{16}$ (D). Direct computation shows $P(X_1^c \mid X) = \frac{1}{8} \neq \frac{3}{16}$ (A false) and $P(X \mid X_2) = \frac{5}{8} \neq \frac{5}{16}$ (C false).
Answer: (B), (D)

Q.14

Tangents to hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to $2x - y = 1$. Points of contact.

Tangent condition for slope $m = 2$: $c^2 = 9(4) - 4 = 32 \Rightarrow c = \pm 4\sqrt{2}$. Matching $y = 2x + c$ to $\frac{xx_0}{9} - \frac{yy_0}{4} = 1$ gives $x_0 = 18/c, y_0 = 4/c$. For $c = 4\sqrt{2}$: $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$; for $c = -4\sqrt{2}$: $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.
Answer: (A), (B)

Q.15

$y' - y\tan x = 2x\sec x$, $y(0) = 0$. *Which value(s) hold?*

Integrating factor $\cos x$ gives $(y\cos x)' = 2x \Rightarrow y\cos x = x^2$ (using $y(0) = 0$), so $y = x^2 \sec x$. Then $y(\pi/4) = \frac{\pi^2}{16}\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$ (A true). Differentiating, $y' = 2x\sec x + y\tan x$; at $x = \pi/3$: $y(\pi/3) = \frac{2\pi^2}{9}$, giving $y'(\pi/3) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$ (D true). Direct computation shows (B) and (C) do not match.

Answer: (A), (D)

Q.16

$f(x) = |x| + |x^2 - 1|$. *Total number of points of local maximum or minimum.*

Breaking at $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$: analysing the sign of f' on each piece shows local minima at $x = -1, 0, 1$ and local maxima at $x = -\frac{1}{2}, \frac{1}{2}$ – five points in total.

Answer: 5

Q.17

$6 + \log_{3/2}\left[\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \cdots}}}\right]$.

Let L satisfy $L = \sqrt{4 - \frac{1}{3\sqrt{2}}L}$; solving the resulting quadratic gives $L = \frac{4\sqrt{2}}{3}$. The full bracketed value is $\frac{1}{3\sqrt{2}} \cdot L = \frac{4}{9} = \left(\frac{3}{2}\right)^{-2}$, so $\log_{3/2}(\cdots) = -2$, giving $6 - 2 = 4$.

Answer: 4

Q.18

$p(x)$ *least-degree real polynomial with local max at* $x = 1$*, local min at* $x = 3$*,* $p(1) = 6, p(3) = 2$*. Find* $p'(0)$*.*

Minimal degree is cubic with $p'(x) = k(x-1)(x-3)$, $k > 0$. Integrating, $p(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + C$. Using $p(3) = C = 2$ and $p(1) = \frac{4k}{3} + 2 = 6$ gives $k = 3$, so $p'(x) = 3(x-1)(x-3)$ and $p'(0) = 9$.

Answer: 9

Q.19

$\vec{a}, \vec{b}, \vec{c}$ *unit vectors with* $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$*. Find* $|2\vec{a} + 5\vec{b} + 5\vec{c}|$*.*

Expanding, $6 - 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 9$, so this dot-product sum $= -\frac{3}{2}$. Then $|\vec{a} + \vec{b} + \vec{c}|^2 = 3 + 2\left(-\frac{3}{2}\right) = 0$, i.e. $\vec{c} = -(\vec{a} + \vec{b})$. Substituting, $2\vec{a} + 5\vec{b} + 5\vec{c} = -3\vec{a}$, so the magnitude is 3.

Answer: 3

Q.20

S *= focus of* $y^2 = 8x$*;* PQ *= common chord of the circle* $x^2 + y^2 - 2x - 4y = 0$ *and the parabola. Area of* $\triangle PQS$*.*

Substituting $x = y^2/8$ into the circle's equation gives $y(y^3 + 48y - 256) = 0$; real roots are $y = 0$ and $y = 4$, giving intersection points $P = (0, 0), Q = (2, 4)$. With focus $S = (2, 0)$, the area is $\frac{1}{2}|0(4 - 0) + 2(0 - 0) + 2(0 - 4)| = 4$.
Answer: 4