QuestIQ Academy

JEE Advanced 2012 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = comprehension-based single correct; Section 3 = one or more correct.

Q.1

Plane through the intersection of $x+2y+3z=2$ *and* $x-y+z=3$*, at distance* $2/\sqrt{3}$ *from* $(3,1,-1)$.

Family: $(1+\lambda)x+(2-\lambda)y+(3+\lambda)z-(2+3\lambda)=0$. Substituting $(3,1,-1)$, the numerator simplifies to -2λ, and the squared denominator to $3\lambda^2+4\lambda+14$. Setting $\dfrac{4\lambda^2}{3\lambda^2+4\lambda+14}=\dfrac{4}{3}$ gives $\lambda=-\frac{7}{2}$, and the resulting plane is $5x-11y+z=17$.
Answer: (A) $5x-11y+z=17$

Q.2

$\triangle PQR$*, area* Δ*,* $a=2, b=\frac{7}{2}, c=\frac{5}{2}$*. Find* $\dfrac{2\sin P-\sin 2P}{2\sin P+\sin 2P}$.

This ratio equals $\dfrac{1-\cos P}{1+\cos P}$. By the cosine rule, $\cos P=\frac{b^2+c^2-a^2}{2bc}=\frac{29}{35}$, giving the ratio $\frac{6/35}{64/35}=\frac{3}{32}$. Using Heron's formula, $\Delta=\sqrt{6}$, and indeed $\left(\frac{3}{4\Delta}\right)^2=\frac{9}{96}=\frac{3}{32}$.
Answer: (C) $(3/(4\Delta))^2$

Q.3

$|\vec{a}+\vec{b}|=\sqrt{29}$, $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$. *Possible value of* $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$.

Writing $\vec{w}=2\hat{i}+3\hat{j}+4\hat{k}$, the condition rearranges to $(\vec{a}+\vec{b})\times\vec{w}=0$, so $\vec{a}+\vec{b}=k\vec{w}$ for some scalar k. Then $|\vec{a}+\vec{b}|=|k|\sqrt{29}=\sqrt{29}\Rightarrow k=\pm1$, giving $(\vec{a}+\vec{b})\cdot(-7,2,3)=k(-14+6+12)=4k=\pm4$.
Answer: (C) 4

Q.4

P *is* 3×3 *with* $P^T=2P+I$*. Find a relation* $PX=?$ *for some nonzero column* X.

Transposing, $P=2P^T+I$; substituting the given relation, $P=2(2P+I)+I=4P+3I\Rightarrow P=-I$. Hence $PX=-X$.

Answer: (D) $PX = -X$

Q.5

$\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$, $a>-1$. *Find* $\lim_{a\to0^+}\alpha(a),\beta(a)$.

Dividing by a and using $\lim_{a\to0}\frac{(1+a)^k-1}{a}=k$, the equation tends to $\frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6}=0$, i.e. $2x^2+3x+1=0=(2x+1)(x+1)$, giving roots $-\frac{1}{2},-1$.
Answer: (B) $-1/2$ and -1

Q.6

Four fair dice $D_1,\ldots,D_4$. Probability D_4 shows a number appearing on at least one of D_1,D_2,D_3.

For any fixed value shown by D_4, $P(\text{none of } D_1,D_2,D_3 \text{ match})=(5/6)^3=125/216$. So the required probability is $1-125/216=91/216$.
Answer: (A) $91/216$

Q.7

$$\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx.$$

The log term is odd and $\cos x$ even, so that part integrates to 0. The remaining $\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx=2\int_0^{\pi/2}x^2\cos x\,dx$; integrating by parts twice gives $2\left[x^2\sin x+2x\cos x-2\sin x\right]_0^{\pi/2}=2\left(\frac{\pi^2}{4}-2\right)=\frac{\pi^2}{2}-4$.
Answer: (B) $\pi^2/2-4$

Q.8

HP with $a_1=5, a_{20}=25$. Least positive integer n with $a_n<0$.

$1/a_n$ is in AP: $\frac{1}{25}=\frac{1}{5}+19d\Rightarrow d=-\frac{4}{475}$. Then $\frac{1}{a_n}=\frac{1}{5}-\frac{4(n-1)}{475}<0\Rightarrow n-1>23.75\Rightarrow n\geq25$.
Answer: (D) 25

Q.9

[Comprehension] a_n = number of n-digit integers from digits $0,1$ with no two consecutive 0's; b_n,c_n count those ending in $1,0$ resp. Find b_6.

$b_n=a_{n-1}$, $c_n=b_{n-1}$, giving $a_n=a_{n-1}+a_{n-2}$ (Fibonacci-type) with $a_1=1,a_2=2$. So $a_1,\ldots,a_5=1,2,3,5,8$, and $b_6=a_5=8$.
Answer: (B) 8

Q.10

[Comprehension, same sequences] Which relation is correct?

Since $a_n = b_n + c_n$ and $b_n = a_{n-1}$ always (the defining recurrence), $a_{17} = a_{16} + a_{15}$ holds by construction.

Answer: (A) $a_{17} = a_{16} + a_{15}$

Q.11

[Comprehension] $f(x) = (1-x)^2 \sin^2 x + x^2$, $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt$, $x \in (1, \infty)$. *Behaviour of* g.

$f(x) > 0$ throughout $(1, \infty)$, so the sign of $g'(x)$ matches $h(x) = \frac{2(x-1)}{x+1} - \ln x$. Since $h(1) = 0$ and numerical testing shows $h(x) < 0$ for all $x > 1$, $g'(x) < 0$ throughout, so g is decreasing on $(1, \infty)$.

Answer: (B) g **is decreasing on** $(1, \infty)$

Q.12

[Comprehension, same f] Statements: P: $\exists x, f(x) + 2x = 2(1 + x^2)$. *Q:* $\exists x, 2f(x) + 1 = 2x(1 + x)$.

Statement P reduces to $-(x-1)^2 \cos^2 x = 1$, impossible since the LHS is always ≤ 0: P is false. For Q, let $\varphi(x) = 2(1-x)^2 \sin^2 x - (2x - 1)$; $\varphi(0) = 1 > 0$ and $\varphi(1) = -1 < 0$, so by the IVT a root exists in $(0, 1)$: Q is true.

Answer: (C) P is false and Q is true

Q.13

[Comprehension] Tangent PT to $x^2 + y^2 = 4$ *at* $P(\sqrt{3}, 1)$*; line* $L \perp PT$ *tangent to* $(x-3)^2 + y^2 = 1$. *A possible equation of* L.

$PT : \sqrt{3}x + y = 4$ (slope $-\sqrt{3}$), so L has slope $1/\sqrt{3}$: write $L : x - \sqrt{3}y = k$. Tangency to the second circle requires $\frac{|3-k|}{2} = 1 \Rightarrow k = 1$ or 5. Taking $k = 1$ gives $x - \sqrt{3}y = 1$.

Answer: (A) $x - \sqrt{3}y = 1$

Q.14

[Comprehension, same circles] A common tangent of the two circles.

The circles are externally tangent ($r_1 + r_2 = 3 =$ distance between centres). Besides the tangent $x = 2$ at the contact point, the two external common tangents satisfy $x\cos\theta + y\sin\theta = 2$ with $|3\cos\theta - 2| = 1$, giving $\cos\theta = \frac{1}{3}$ and the line $x + 2\sqrt{2}y = 6$.

Answer: (D) $x + 2\sqrt{2}y = 6$

Q.15

$f(x) = a_n + \sin \pi x$ on $[2n, 2n+1]$, $f(x) = b_n + \cos \pi x$ on $(2n-1, 2n)$, for all integers n; f continuous. Which hold?

Matching at $x = 2n$: $b_n + \cos(2n\pi) = a_n + \sin(2n\pi) \Rightarrow a_n - b_n = 1$. Matching at $x = 2n-1$: $a_{n-1} + \sin((2n-1)\pi) = b_n + \cos((2n-1)\pi) \Rightarrow a_{n-1} - b_n = -1$.
Answer: (B), (D)

Q.16

$f(x) = \int_0^x e^{t^2}(t-2)(t-3)\,dt$, $x \in (0, \infty)$. Which hold?

By the FTC, $f'(x) = e^{x^2}(x-2)(x-3)$, positive on $(0, 2)$, negative on $(2, 3)$, positive on $(3, \infty)$: local max at $x = 2$, decreasing on $(2, 3)$, local min at $x = 3$. Since $f'(2) = f'(3) = 0$, Rolle's theorem guarantees some $c \in (2, 3)$ with $f''(c) = 0$.
Answer: (A), (B), (C), (D)

Q.17

Lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar. Plane(s) containing both.

Coplanarity (triple product with connecting vector $(2, 0, 0)$) gives $2(k^2 - 4) = 0 \Rightarrow k = \pm 2$. For $k = 2$: normal $\propto (0, 1, -1)$, plane $y - z = -1$. For $k = -2$: normal $\propto (0, 1, 1)$, plane $y + z = -1$.
Answer: (B), (C)

Q.18

$P(X \mid Y) = \frac{1}{2}$, $P(Y \mid X) = \frac{1}{3}$, $P(X \cap Y) = \frac{1}{6}$. Which hold?

From the conditionals, $P(Y) = \frac{1}{3}$ and $P(X) = \frac{1}{2}$. Then $P(X \cup Y) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$ (A). Since $P(X)P(Y) = \frac{1}{6} = P(X \cap Y)$, X, Y are independent (B), ruling out (C); and $P(X^c \cap Y) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6} \neq \frac{1}{3}$, ruling out (D).
Answer: (A), (B)

Q.19

If the adjoint of a 3×3 matrix P is $\begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$, find possible $\det P$.

Using $\det(\operatorname{adj} P) = (\det P)^2$ for a 3×3 matrix: $\det(\operatorname{adj} P) = 1(3-7) - 4(6-7) + 4(2-1) = -4 + 4 + 4 = 4$, so $(\det P)^2 = 4 \Rightarrow \det P = \pm 2$.
Answer: (A), (D)

Q.20

$f : (-1, 1) \to \mathbb{R}$, $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2 \theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Find $f(1/3)$.

Solving $\cos 4\theta = \frac{1}{3}$ via $8u^2 - 8u + \frac{2}{3} = 0$ ($u = \cos^2\theta$) gives two valid roots depending on the branch: for $\theta \in (0, \pi/4)$, $\sec^2\theta = 6 - 2\sqrt{6}$, giving $f(1/3) = 1 + \sqrt{3/2}$; for $\theta \in (\pi/4, \pi/2)$, $\sec^2\theta = 6 + 2\sqrt{6}$, giving $f(1/3) = 1 - \sqrt{3/2}$. Since these two branches produce two different values for the same input, f is not well-defined by the given data – this is exactly why the official key awarded zero marks to all candidates on this question. Both candidate values match options (A) and (B) respectively.

Answer: Question voided (official key: zero marks to all); candidate values $1 - \sqrt{3/2}$ (A) and $1 + \sqrt{3/2}$ (B) both arise, confirming the ambiguity