QuestIQ Academy

JEE Advanced 2013 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = single-digit integer.

Q.1

Complex numbers α and $1/\bar{\alpha}$ lie on circles $|z - z_0| = r$ and $|z - z_0| = 2r$ resp. If $z_0 = x_0 + iy_0$ satisfies $2|z_0|^2 = r^2 + 2$, find $|\alpha|$.

Let $\rho = |\alpha|$, $p = z_0$, $S = \mathrm{Re}(\alpha\bar{p})$. From $|\alpha - p|^2 = r^2$: $|p|^2 = r^2 - \rho^2 + 2S$. Since $1/\bar{\alpha} = \alpha/\rho^2$, from $|\alpha/\rho^2 - p|^2 = 4r^2$: $|p|^2 = 4r^2 - 1/\rho^2 + 2S/\rho^2$. Equating and using $2|p|^2 = r^2 + 2$ to eliminate S leads, after simplification, to $r^2 = 7r^2\rho^2$, so $\rho^2 = 1/7$.
Answer: (C) $1/\sqrt{7}$

Q.2

Four persons independently solve a problem with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Probability at least one solves it.

$P(\text{none}) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{7}{8}\right) = \frac{21}{256}$. So $P(\text{at least one}) = 1 - \frac{21}{256} = \frac{235}{256}$.
Answer: (A) $235/256$

Q.3

$f : [\frac{1}{2}, 1] \to \mathbb{R}$ positive, non-constant, differentiable, $f'(x) < 2f(x)$, $f(\frac{1}{2}) = 1$. Interval containing $\int_{1/2}^{1} f\,dx$.

Let $g(x) = f(x)e^{-2x}$; $g'(x) = e^{-2x}(f' - 2f) < 0$, so g is strictly decreasing. Then $g(1) < g(\frac{1}{2}) = e^{-1}$, giving $f(1) < e$, and by continuity $f(x) < e^{2x-1}$ on $(\frac{1}{2}, 1]$. Hence $\int_{1/2}^{1} f\,dx < \int_{1/2}^{1} e^{2x-1}dx = \frac{e-1}{2}$; positivity of f gives the integral > 0.
Answer: (D) $\left(0, \frac{e-1}{2}\right)$

Q.4

Number of real roots of $x^2 - x\sin x - \cos x = 0$.

Let $g(x) = x^2 - x\sin x - \cos x$; g is even (check $g(-x) = g(x)$), $g(0) = -1$. For $x > 0$, $g'(x) = 2x - x\cos x = x(2-\cos x) > 0$, so g increases from -1 to ∞ on $(0, \infty)$ – exactly one positive root, and by symmetry one negative root.
Answer: (C) 2

Q.5

Area enclosed by $y = \sin x + \cos x$ *and* $y = |\cos x - \sin x|$ *on* $[0, \frac{\pi}{2}]$.

On $[0, \frac{\pi}{4}]$, $|\cos x - \sin x| = \cos x - \sin x$, difference $= 2\sin x$, integral $= 2 - \sqrt{2}$. On $[\frac{\pi}{4}, \frac{\pi}{2}]$, difference $= 2\cos x$, integral $= 2 - \sqrt{2}$. Total area $= 4 - 2\sqrt{2} = 2\sqrt{2}(\sqrt{2} - 1)$.
Answer: (B) $2\sqrt{2}(\sqrt{2} - 1)$

Q.6

Curve through $(1, \frac{\pi}{6})$ *with slope* $\frac{y}{x} + \sec(\frac{y}{x})$. *Find its equation.*

Let $v = y/x$; then $xv' = \sec v \Rightarrow \cos v\, dv = dx/x \Rightarrow \sin v = \ln x + C$. At $x = 1, v = \frac{\pi}{6}$: $C = \frac{1}{2}$. So $\sin(y/x) = \ln x + \frac{1}{2}$.
Answer: (A) $\sin(y/x) = \log x + \frac{1}{2}$

Q.7

Value of $\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)$.

$\sum_{k=1}^{n} 2k = n(n+1)$, and $\cot^{-1}(1 + n(n+1)) = \tan^{-1}(n+1) - \tan^{-1} n$ (telescoping identity). Summing $n = 1$ to 23 gives $\tan^{-1} 24 - \tan^{-1} 1 = \tan^{-1}\frac{23}{25}$, so cot of this is $25/23$.
Answer: (B) $25/23$

Q.8

For $a > b > c > 0$, *distance between* $(1, 1)$ *and intersection of* $ax + by + c = 0$, $bx + ay + c = 0$ *is less than* $2\sqrt{2}$. *Which condition follows?*

Subtracting the lines gives $x = y = -c/(a+b)$. Distance$^2 = 2\left(1 + \frac{c}{a+b}\right)^2 < 8 \Rightarrow -3 < \frac{c}{a+b} < 1$; since $c/(a+b) > 0$, this reduces to $c < a+b$, i.e. $a + b - c > 0$. (Due to a printing ambiguity in the option list, the official key accepted both the equivalent-labelled options.)
Answer: (A) $a + b - c > 0$ **(official key: A or C)**

Q.9

Feet of perpendiculars from points on line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ *to plane* $x + y + z = 3$ *lie on which line?*

Parametrize the line as $(2t-2, -t-1, 3t)$ and drop the foot onto the plane using $F = P - \frac{(P\cdot n - 3)}{|n|^2} n$ with $n = (1, 1, 1)$. This gives $F = \left(\frac{2t}{3}, \frac{3-7t}{3}, \frac{5t+6}{3}\right)$, tracing the line through $(0, 1, 2)$ with direction $(2, -7, 5)$: $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$.
Answer: (D)

Q.10

$\vec{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ are diagonals of parallelogram PQRS; $\vec{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$. Volume of parallelepiped by $\vec{PT}, \vec{PQ}, \vec{PS}$.

With $\vec{PQ} = u, \vec{PS} = v$: $u + v = \vec{PR}$, $u - v = \vec{SQ}$, so $u = 2\hat{i} - \hat{j} - 3\hat{k}$, $v = \hat{i} + 2\hat{j} + \hat{k}$. Then $u \times v = 5\hat{i} - 5\hat{j} + 5\hat{k}$, and $\vec{PT} \cdot (u \times v) = 5 - 10 + 15 = 10$.

Answer: (C) 10

Q.11

$S_n = \sum_{k=1}^{4n} (-1)^{k(k+1)/2} k^2$. Possible value(s).

Grouping in blocks of four ($k = 4m-3, \ldots, 4m$ with signs $-, -, +, +$) gives each block summing to $32m - 12$, so $S_n = \sum_{m=1}^{n} (32m - 12) = 16n^2 + 4n = 4n(4n + 1)$. Testing $n = 8$: $S_8 = 1056$; testing $n = 9$: $S_9 = 1332$.

Answer: (A) 1056, (D) 1332

Q.12

For 3×3 matrices M, N, which statement(s) is/are NOT correct?

$(N^T MN)^T = \pm N^T MN$ according as M is symmetric/skew – always true (A is a true statement, so not the answer). $MN - NM$ is always skew-symmetric – true (B is not the answer). $(MN)^T = NM \neq MN$ in general, so "MN symmetric for all symmetric M, N" is false (C is incorrect, hence an answer). The correct identity is $\mathrm{adj}(MN) = \mathrm{adj}(N)\,\mathrm{adj}(M)$, not $\mathrm{adj}(M)\,\mathrm{adj}(N)$, so (D) is also incorrect.

Answer: (C), (D)

Q.13

$f(x) = x \sin \pi x$, $x > 0$. Where does $f'(x)$ vanish?

By Rolle's theorem, since $f(n) = f(n + 1) = 0$ for integers n, a critical point exists in every $(n, n + 1)$. Setting $t = x - n \in (0, 1)$, the equation $f'(x) = 0$ becomes $\tan(\pi t) = -\pi(n + t)$; since the RHS is negative, a solution requires $\tan(\pi t) < 0$, i.e. $t \in (\frac{1}{2}, 1)$, where $\tan(\pi t)$ increases monotonically from $-\infty$ to 0 – exactly one root there.

Answer: (B), (C)

Q.14

Rectangular sheet, sides in ratio $8 : 15$, fixed perimeter; open box formed by removing equal corner squares of total area 100; resulting box has maximum volume. Find the sheet's side lengths.

Let sides be $8k, 15k$, corner squares of side x; $4x^2 = 100 \Rightarrow x = 5$. Volume $V(x) = x(8k - 2x)(15k - 2x)$; setting $V'(5) = 0$ gives $6k^2 - 23k + 15 = 0$, so $k = 3$ or $k = 5/6$ (rejected, makes width negative). With $k = 3$: sides are 24 and 45.

Answer: (A) 24, (C) 45

Q.15

Line l through the origin, perpendicular to l_1 (dir. $(1,2,2)$) and l_2 (dir. $(2,2,1)$). Point(s) on l_2 at distance $\sqrt{17}$ from $l \cap l_1$.

Direction of l is $(1,2,2) \times (2,2,1) = (-2,3,-2)$. Solving $l \cap l_1$ gives the point $(2,-3,2)$. Points on l_2 at distance $\sqrt{17}$ from this satisfy $9s^2 + 28s + 20 = 0$, giving $s = -2$ (point $(-1,-1,0)$) or $s = -\frac{10}{9}$ (point $(\frac{7}{9}, \frac{7}{9}, \frac{8}{9})$).

Answer: (B) $(-1,-1,0)$**, (D)** $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

Q.16

$(1+x)^{n+5}$ *has three consecutive coefficients in ratio* $5:10:14$. *Find* n.

Using $\frac{C(N,r)}{C(N,r-1)} = \frac{N-r+1}{r}$ with $N = n+5$: the two consecutive ratios give $N+1 = 3r$ and $12r = 5N - 7$. Solving simultaneously gives $N = 11$, so $n = 6$.

Answer: 6

Q.17

n cards numbered 1 to n; two consecutive cards removed leave sum 1224. If smaller removed number is k, find $k - 20$.

$\frac{n(n+1)}{2} - (2k+1) = 1224$. Trying $n = 50$: $\frac{50 \cdot 51}{2} = 1275$, so $2k+1 = 51 \Rightarrow k = 25$ (valid since $1 \le k \le 49$). Thus $k - 20 = 5$.

Answer: 5

Q.18

Independent events E_1, E_2, E_3; only-E_1 prob. α, only-E_2 prob. β, only-E_3 prob. γ; none-occur prob. p satisfies $(\alpha - 2\beta)p = \alpha\beta$, $(\beta - 3\gamma)p = 2\beta\gamma$. Find $P(E_1)/P(E_3)$.

With $x, y, z = P(E_1), P(E_2), P(E_3)$ and $A = \frac{x}{1-x}, B = \frac{y}{1-y}, C = \frac{z}{1-z}$, the given equations become $A - 2B = AB$ and $B - 3C = 2BC$, giving $B = \frac{A}{2+A}$ and $C = \frac{A}{6+5A}$. Then $\frac{x}{z} = \frac{A(1+C)}{C(1+A)}$ simplifies, after substitution, to exactly 6 (all dependence on A cancels).

Answer: 6

Q.19

Vertical line through $(h,0)$ meets ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at P, Q; tangents at P, Q meet at R. $\Delta(h) =$ area of $\triangle PQR$; $\Delta_1 = \max_{[1/2,1]} \Delta(h)$, $\Delta_2 = \min_{[1/2,1]} \Delta(h)$. Find $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2$.

Adding the tangents at $P = (h, y_0), Q = (h, -y_0)$ gives $R = (4/h, 0)$, and $\Delta(h) = y_0 \cdot \frac{4-h^2}{h} = \frac{\sqrt{3}}{2} \cdot \frac{(4-h^2)^{3/2}}{h}$, which is decreasing on $[\frac{1}{2}, 1]$. So $\Delta_1 = \Delta(\frac{1}{2}) = \frac{45\sqrt{5}}{8}$ and $\Delta_2 = \Delta(1) = \frac{9}{2}$. Then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$.

Answer: 9

Q.20

Set of eight vectors $V = \{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\}$. *Three non-coplanar vectors can be chosen from* V *in* 2^p *ways. Find* p.

Total triples: $\binom{8}{3} = 56$. A triple containing an antipodal pair $(v, -v)$ is automatically coplanar; there are 4 antipodal pairs, each combinable with any of the remaining 6 vectors, giving 24 coplanar triples (direct determinant checks confirm no other coplanar triples occur). So non-coplanar triples $= 56 - 24 = 32 = 2^5$.

Answer: 5