QuestIQ Academy

JEE Advanced 2013 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = one or more correct; Section 2 = comprehension-based single correct; Section 3 = matching-list single correct.

Q.1

$w = \frac{\sqrt{3}+i}{2}$, $P = \{w^n : n = 1, 2, \dots\}$. $H_1 = \{\mathrm{Re}\, z > \frac{1}{2}\}$, $H_2 = \{\mathrm{Re}\, z < -\frac{1}{2}\}$. *For* $z_1 \in P \cap H_1, z_2 \in P \cap H_2$, *find* $\angle z_1 O z_2$.

$w = e^{i\pi/6}$, so w^n lies at angle $30n^\circ$. $\mathrm{Re}(w^n) > \frac{1}{2}$ for $n = 1, 11, 12$ (angles $30^\circ, 330^\circ, 0^\circ$); $\mathrm{Re}(w^n) < -\frac{1}{2}$ for $n = 5, 6, 7$ (angles $150^\circ, 180^\circ, 210^\circ$). Checking all pairings, the angle between z_1, z_2 takes values $120^\circ = \frac{2\pi}{3}$ or $150^\circ = \frac{5\pi}{6}$ (and π, not offered).

Answer: (C) $2\pi/3$, (D) $5\pi/6$

Q.2

If $3^x = 4^{x-1}$, *find* x.

$x \ln 3 = (x-1)\ln 4 \Rightarrow x = \dfrac{\ln 4}{\ln 4 - \ln 3} = \dfrac{2\ln 2}{2\ln 2 - \ln 3}$. Converting to each base confirms options (A) $2\log_3 2/(2\log_3 2 - 1)$, (B) $2/(2 - \log_2 3)$, and (C) $1/(1 - \log_4 3)$ are all algebraically equal to this value; (D) reduces to a different expression.

Answer: (A), (B), (C)

Q.3

$\omega \neq 1$ *complex cube root of unity,* $P = [p_{ij}]$, $n \times n$, $p_{ij} = \omega^{i+j}$. *Find* n *for which* $P^2 \neq 0$.

$P = vv^T$ with $v_i = \omega^i$, so $P^2 = (v^T v)P$ where $v^T v = \sum_{i=1}^{n} \omega^{2i}$. This sum is 0 exactly when $n \equiv 0 \pmod 3$ (grouping terms in blocks of 3, each block sums to 0), and nonzero otherwise. So $P^2 \neq 0$ iff $3 \nmid n$: true for $55, 56, 58$, false for 57.

Answer: (B) 55, (C) 58, (D) 56

Q.4

$f(x) = 2|x| + |x+2| - \big||x+2| - 2|x|\big|$. *Where does* f *have a local min or max?*

Using $a+b-|a-b| = 2\min(a,b)$ with $a = 2|x|, b = |x+2|$: $f(x) = 2\min(2|x|, |x+2|)$. The two expressions cross at $x = -2$ (vertex of $|x+2|$) and $x = -\frac{2}{3}$ (intersection point); checking slopes on either side shows a local minimum at $x = -2$ (slope $-1 \to +1$) and a local maximum at $x = -\frac{2}{3}$ (slope $+1 \to -2$); $x = 2$ is only a slope-kink within a monotone increasing stretch.
Answer: (A) -2, (B) $-2/3$

Q.5

$\lim_{n\to\infty} \dfrac{1^a + 2^a + \cdots + n^a}{(n+1)^{a-1}\big[(na+1) + \cdots + (na+n)\big]} = \dfrac{1}{60}$. *Find a.*

For large n, numerator $\sim \frac{n^{a+1}}{a+1}$ and denominator $\sim n^{a-1} \cdot n^2(a+\frac{1}{2})$, giving the limit $= \dfrac{1}{(a+1)(a+1/2)} = \dfrac{1}{60}$, i.e. $2a^2 + 3a - 119 = 0$, so $a = 7$ or $a = -8.5$. Only $a = 7$ is consistent with the asymptotic growth used (for $a < -1$ the numerator sum instead converges).
Answer: (B) 7

Q.6

Circle(s) touching the x-axis at distance 3 from the origin, with y-intercept of length $2\sqrt{7}$.

Circle centered $(3, k)$, radius $|k|$: $x^2 + y^2 - 6x - 2ky + 9 = 0$. Setting $x = 0$ gives $y^2 - 2ky + 9 = 0$ with root-difference $2\sqrt{k^2 - 9} = 2\sqrt{7} \Rightarrow k^2 = 16 \Rightarrow k = \pm 4$, giving $x^2 + y^2 - 6x \mp 8y + 9 = 0$.
Answer: (A), (C)

Q.7

Lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Find possible α.

The coplanarity (scalar triple product) condition reduces to $(\alpha - 5)[(3-\alpha)(2-\alpha) - 2] = 0$, i.e. $\alpha^2 - 5\alpha + 4 = 0$, giving $\alpha = 1$ or $\alpha = 4$.
Answer: (A) 1, (D) 4

Q.8

Triangle PQR, P largest angle, $\cos P = \frac{1}{3}$; incircle touches PQ, QR, RP with tangent lengths PN, QL, RM consecutive even integers. Possible side length(s)?

Writing tangent lengths as $2t, 2t+2, 2t+4$ from P, Q, R (smallest from P, the largest-angle vertex), the sides work out to $QR = 4t + 6$, $RP = 4t + 4$, $PQ = 4t + 2$. Substituting into $\cos P = \frac{RP^2 + PQ^2 - QR^2}{2 \cdot RP \cdot PQ} = \frac{1}{3}$ gives $t^2 - 3t - 4 = 0 \Rightarrow t = 4$, so the sides are $18, 20, 22$.
Answer: (B) 18, (D) 22

Q.9

[Comprehension] $S = S_1 \cap S_2 \cap S_3$, $S_1 = \{|z| < 4\}$, $S_2 = \{\mathrm{Im}[(z - 1 + \sqrt{3}i)/(1 - \sqrt{3}i)] > 0\}$, $S_3 = \{\mathrm{Re}\, z > 0\}$. Area of S.

Simplifying, S_2 is the half-plane $\sqrt{3}x + y > 0$, i.e. angles in $(-60°, 120°)$. Intersected with S_3 (angles in $(-90°, 90°)$) gives the sector $(-60°, 90°)$, a $150° = \frac{5\pi}{6}$ sector of the disk of radius 4: area $= \frac{1}{2}(16)\left(\frac{5\pi}{6}\right) = \frac{20\pi}{3}$.
Answer: (B) $20\pi/3$

Q.10

[Comprehension, same S] $\min_{z\in S} |1 - 3i - z|$.

The point $(1, -3)$ has angle $\approx -71.6°$, just outside the sector's lower boundary ray at $-60°$. The nearest point of S is the foot of perpendicular onto that ray, giving minimum distance $\sqrt{10 - t^{*2}}$ with $t^* = (1 + 3\sqrt{3})/2$, which simplifies exactly to $\frac{3-\sqrt{3}}{2}$.
Answer: (C) $(3 - \sqrt{3})/2$

Q.11

[Comprehension] Boxes $B_1(1w, 3r, 2b)$, $B_2(2w, 3r, 4b)$, $B_3(3w, 4r, 5b)$. Draw one ball from each; probability all three are the same colour.

$P(\text{all white}) = \frac{1}{6}\cdot\frac{2}{9}\cdot\frac{1}{4}$, $P(\text{all red}) = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}$, $P(\text{all black}) = \frac{1}{3}\cdot\frac{4}{9}\cdot\frac{5}{12}$. Summing over a common denominator of 324 gives $\frac{3+18+20}{324} = \frac{41}{324} = \frac{82}{648}$.
Answer: (A) 82/648

Q.12

[Comprehension, same boxes] Two balls drawn (no replacement) from a randomly chosen box; one white, one red. Probability they came from B_2.

$P(WR \mid B_1) = \frac{1\cdot 3}{\binom{6}{2}} = \frac{1}{5}$, $P(WR \mid B_2) = \frac{2\cdot 3}{\binom{9}{2}} = \frac{1}{6}$, $P(WR \mid B_3) = \frac{3\cdot 4}{\binom{12}{2}} = \frac{2}{11}$. By Bayes' theorem with equal priors, $P(B_2 \mid WR) = \dfrac{1/6}{1/5 + 1/6 + 2/11} = \dfrac{55}{181}$.
Answer: (D) 55/181

Q.13

[Comprehension] $f : [0, 1] \to \mathbb{R}$ twice differentiable, $f(0) = f(1) = 0$, $f''(x) - 2f'(x) + f(x) \geq e^x$. Which holds for $0 < x < 1$?

Let $g(x) = e^{-x}f(x)$; then $g''(x) = e^{-x}(f'' - 2f' + f) \geq 1$. Setting $h(x) = g(x) - \frac{x^2-x}{2}$ gives $h'' \geq 0$ with $h(0) = h(1) = 0$, so convexity forces $h(x) \leq 0$, i.e. $g(x) \leq -\frac{x(1-x)}{2} < 0$ on $(0, 1)$. Since $e^{-x} > 0$, this means $f(x) < 0$ throughout.
Answer: (D) $-\infty < f(x) < 0$

Q.14

[Comprehension, same f] If $e^{-x}f(x)$ attains its minimum on $[0, 1]$ at $x = \frac{1}{4}$, which holds?

With $g(x) = e^{-x}f(x)$ strictly convex (from the previous part), the interior minimum at $x = \frac{1}{4}$ means $g' < 0$ for $x < \frac{1}{4}$ and $g' > 0$ for $x > \frac{1}{4}$. Since $g'(x) = e^{-x}(f'(x) - f(x))$, this translates to $f'(x) < f(x)$ for $0 < x < \frac{1}{4}$.
Answer: (C) $f'(x) < f(x),\ 0 < x < 1/4$

Q.15

[Comprehension] PQ is a focal chord of $y^2 = 4ax$; tangents at P, Q meet on $y = 2x + a, a > 0$. Length of PQ.

With P, Q parametrized by t_1, t_2 ($t_1t_2 = -1$ for a focal chord), the tangents meet at $(-a, a(t_1 + t_2))$; lying on $y = 2x + a$ forces $t_1 + t_2 = -1$. Then $(t_1 - t_2)^2 = (t_1 + t_2)^2 - 4t_1t_2 = 5$, and $PQ^2 = a^2(t_1 - t_2)^2[(t_1 + t_2)^2 + 4] = 25a^2$, so $PQ = 5a$.
Answer: (B) $5a$

Q.16

[Comprehension, same chord] If PQ subtends angle θ at the vertex, find $\tan\theta$.

With $t_1 + t_2 = -1, t_1t_2 = -1$: $\tan\theta = \dfrac{|\vec{OP} \times \vec{OQ}|}{\vec{OP} \cdot \vec{OQ}} = \dfrac{2a^2|t_1 - t_2|}{a^2[(t_1t_2)^2 + 4t_1t_2]} = \dfrac{2a^2\sqrt{5}}{-3a^2} = -\dfrac{2}{3}\sqrt{5}$.
Answer: (D) $-\frac{2}{3}\sqrt{5}$

Q.17

[Matching list] Line $L : y = mx+3$ meets y-axis at $E(0, 3)$ and the parabola $y^2 = 16x$ $(0 \le y \le 6)$ at $F(x_0, y_0)$; tangent at F meets y-axis at $G(0, y_1)$. m chosen so area of $\triangle EFG$ has a local maximum. Match m, max area, y_0, y_1.

The tangent gives $y_1 = y_0/2$; area $= \frac{y_0^2}{32}(3 - \frac{y_0}{2})$ for $0 \le y_0 \le 6$, maximized at $y_0 = 4$ (from $A'(y_0) = 0$). Then $x_0 = 1$, $m = (y_0 - 3)/x_0 = 1$, max area $= \frac{1}{2}$, $y_1 = 2$.
Answer: $m = 1$, **area** $= \frac{1}{2}$, $y_0 = 4$, $y_1 = 2$ **(P→4, Q→1, R→2, S→3 – option A)**

Q.18

[Matching list] Match four trigonometric/inverse-trig identities to their values.

(P) Using $\theta = \tan^{-1} y, \varphi = \sin^{-1} y$: $\cos\theta + y\sin\theta = \sqrt{1 + y^2}$ and $\cot\varphi + \tan\varphi = \frac{1}{y\sqrt{1-y^2}}$; the whole bracketed expression simplifies to exactly 1, so P → 1. (Q) $\cos x + \cos y + \cos z = \sin x + \sin y + \sin z = 0$ forces the unit vectors 120° apart, giving $\cos\frac{x-y}{2} = \cos 60° = \frac{1}{2}$, so Q→ $\frac{1}{2}$. (R) Testing $x = \pi/4$ satisfies the given trig identity, giving $\sec x = \sqrt{2}$, so R→ $\sqrt{2}$. (S) Solving $\cot(\sin^{-1}\sqrt{1 - x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$ gives $x^2 = 5/12$, i.e. $x = \frac{1}{2}\sqrt{5/3}$, so S→ $\frac{1}{2}\sqrt{5/3}$.
Answer: P→4, Q→3, R→2, S→1 (option B)

Q.19

[Matching list] Lines L_1, L_2 and planes $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$; plane $ax + by + cz = d$ passes through $L_1 \cap L_2$ and is perpendicular to P_1, P_2. Match a, b, c, d.

$L_1 \cap L_2 = (5, -2, -1)$. Perpendicularity to both planes means the required normal is $n_1 \times n_2 = (7, 1, 2) \times (3, 5, -6) = (-16, 48, 32) \parallel (1, -3, -2)$. The plane through $(5, -2, -1)$ with this normal is $x - 3y - 2z = 13$, so $(a, b, c, d) = (1, -3, -2, 13)$.
Answer: P→3, Q→2, R→4, S→1 (option A)

Q.20

[Matching list] Volume/area scaling identities for vectors $\vec{a}, \vec{b}, \vec{c}$.

(P) Using $[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}] = [\vec{a}, \vec{b}, \vec{c}]^2$: with $V = 2$, scaling by $2, 3, 1$ gives $6 \times 4 = 24$. (Q) Using $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$: with $V = 5$, scaling by $3, 1, 2$ gives $6 \times 10 = 60$. (R) $(2\vec{a} + 3\vec{b}) \times (\vec{a} - \vec{b}) = -5(\vec{a} \times \vec{b})$, so the new triangle area is $5 \times 40/2 = 100$. (S) $(\vec{a} + \vec{b}) \times \vec{a} = -(\vec{a} \times \vec{b})$, so the new parallelogram area is unchanged at 30.
Answer: P→3, Q→4, R→1, S→2 (option C)