QuestIQ Academy

JEE Advanced 2014 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = one or more correct; Section 2 = single-digit integer.

Q.1

M, N are 3×3 matrices with $MN = NM$, $M \neq N^2$, $M^2 = N^4$. Which is (are) true about $M^2 + MN^2$?

Since $MN = NM$, $M^2 - (N^2)^2 = (M - N^2)(M + N^2) = 0$. As $M \neq N^2$, $M - N^2 \neq 0$; since a nonzero matrix times another gives zero only if the second is singular, $\det(M + N^2) = 0$. So $\det(M^2 + MN^2) = \det(M)\det(M + N^2) = 0$ (true (A), false (C)). Also there is a nonzero vector v with $(M + N^2)v = 0$, giving $(M^2 + MN^2)v = 0$ for a nonzero matrix U built from v (true (B), false (D)).

Answer: (A), (B)

Q.2

$f, g : [0,1] \to \mathbb{R}$ continuous with $\max f = \max g$ on $[0,1]$. Which equality must hold for some $c \in [0,1]$?

Let c_f, c_g be points where f, g attain the common maximum M. Then $f(c_f) - g(c_f) \geq 0$ and $f(c_g) - g(c_g) \leq 0$, so by the IVT there is c with $f(c) = g(c)$. For any single function φ applied identically to both sides (as in (A): $\varphi(t) = t^2 + 3t$, and (D): $\varphi(t) = t^2$), this common value automatically gives $\varphi(f(c)) = \varphi(g(c))$. Options (B),(C) apply different functions to each side and are not guaranteed.

Answer: (A), (D)

Q.3

$f(x) = \displaystyle\int_{1/x}^{x} e^{-(t+1/t)}\frac{dt}{t}$, $x > 0$. Which is (are) true?

By Leibniz's rule, $f'(x) = \dfrac{e^{-(x+1/x)}}{x} + \dfrac{e^{-(x+1/x)}}{x} = \dfrac{2e^{-(x+1/x)}}{x} > 0$ for all $x > 0$, so f is increasing on all of $(0,\infty)$ – true on $[1,\infty)$ (A), false that it decreases on $(0,1)$ (B). Swapping the limits of integration gives $f(1/x) = -f(x)$, so $f(x) + f(1/x) = 0$ (true (C)). Then $f(2^{-x}) = -f(2^x)$, so $F(x) = f(2^x)$ satisfies $F(-x) = -F(x)$, i.e. odd (true (D)).

Answer: (A), (C), (D)

Q.4

$f(x) = x^5 - 5x + a$. Root count for various ranges of a.

$f'(x) = 5(x^2-1)$, critical points $x = \pm 1$; local max $f(-1) = a+4$, local min $f(1) = a-4$. Three real roots occur exactly when the local max is positive and the local min is negative: $a + 4 > 0$ and $a - 4 < 0$, i.e. $-4 < a < 4$ (true (D), false (C)). For $a > 4$, the local min is already positive, so only one real root (true (B)); for $a < -4$ the local max is already negative, giving one root, not three (false (A)).

Answer: (B), (D)

Q.5

$f : [a, b] \to [1, \infty)$ continuous; $g(x) = 0$ $(x < a)$, $\int_a^x f(t)dt$ $(a \le x \le b)$, $\int_a^b f(t)dt$ $(x > b)$. Which is (are) true?

At $x = a$: both one-sided limits of g equal 0 (continuous), but the left derivative is 0 while the right derivative is $f(a) \ge 1$ – not differentiable (true (A)). At $x = b$: g is continuous (both sides give $\int_a^b f$), but the left derivative is $f(b) \ge 1$ while the right derivative is 0 – not differentiable (true (C)). This rules out (B) (fails at both points) and (D) (fails at both, not just one).

Answer: (A), (C)

Q.6

$f(x) = [\log(\sec x + \tan x)]^3$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Which is (are) true?

Since $\sec(-x)+\tan(-x) = \sec x - \tan x = 1/(\sec x + \tan x)$, $u(x) = \log(\sec x + \tan x)$ is odd, so $f = u^3$ is also odd (true (A), false (D)). Since $u'(x) = \sec x > 0$ throughout, u is strictly increasing, hence so is $f = u^3$ (composition of increasing functions) – one-one (true (B)). As $x \to \pm\pi/2$, $u \to \pm\infty$ so $f \to \pm\infty$; by continuity and the IVT, f is onto $\mathbb{R}$ (true (C)).

Answer: (A), (B), (C)

Q.7

$P(\lambda, \lambda, \lambda)$; $PQ \perp$ line $y = x, z = 1$; $PR \perp$ line $y = -x, z = -1$; $\angle QPR = 90°$. Find possible λ.

The foot of perpendicular from P to $(t, t, 1)$ occurs at $t = \lambda$, giving $Q = (\lambda, \lambda, 1)$; to $(s, -s, -1)$ occurs at $s = 0$, giving $R = (0, 0, -1)$. Then $\vec{PQ} = (0, 0, 1-\lambda)$ and $\vec{PR} = (-\lambda, -\lambda, -1-\lambda)$; their dot product is $(1-\lambda)(-1-\lambda) = \lambda^2 - 1$. Setting this to 0 gives $\lambda = \pm 1$; but $\lambda = 1$ makes $\vec{PQ} = \vec{0}$ (degenerate, $Q = P$), so only $\lambda = -1$ is valid.

Answer: (C)

Q.8

$\vec{x}, \vec{y}, \vec{z}$ each of magnitude $\sqrt{2}$, pairwise angle $\pi/3$. $\vec{a} \neq 0$, $\vec{a} \perp \vec{x}$ and $\vec{a} \perp (\vec{y} \times \vec{z})$; $\vec{b} \neq 0$, $\vec{b} \perp \vec{y}$ and $\vec{b} \perp (\vec{z} \times \vec{x})$. Which is (are) true?

Since $\vec{a} \perp (\vec{y} \times \vec{z})$, $\vec{a}$ lies in the span of $\vec{y}, \vec{z}$: $\vec{a} = \alpha\vec{y} + \beta\vec{z}$. Using $\vec{y} \cdot \vec{x} = \vec{z} \cdot \vec{x} = |\vec{y}||\vec{x}| \cos(\pi/3) = 1$, the condition $\vec{a} \cdot \vec{x} = 0$ gives $\alpha + \beta = 0$, so $\vec{a} = \alpha(\vec{y} - \vec{z})$; then $\vec{a} \cdot \vec{y} = \alpha(2 - 1) = \alpha$, giving $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$ (true (B), false (D)). By the same argument, $\vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$ (true (A)). Then $\vec{a} \cdot \vec{b} = (\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})(\vec{y} - \vec{z}) \cdot (\vec{z} - \vec{x})$, and $(\vec{y} - \vec{z}) \cdot (\vec{z} - \vec{x}) = 1 - 1 - 2 + 1 = -1$, giving $\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$ (true (C)).
Answer: (A), (B), (C)

Q.9

Circle S through $(0, 1)$, orthogonal to $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Which is (are) true?

With $S : x^2 + y^2 + 2gx + 2fy + c = 0$: passing through $(0, 1)$ gives $2f + c = -1$. Orthogonality with $x^2 + y^2 - 1 = 0$ gives $c = 1$ (so $f = -1$). Orthogonality with $x^2 + y^2 - 2x - 15 = 0$ gives $-2g = 1 - 15 = -14$, so $g = 7$. Thus centre $= (-7, 1)$ (true (C)) and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{49 + 1 - 1} = 7$ (true (B)).
Answer: (B), (C)

Q.10

M a 2×2 symmetric integer matrix, $M = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Which condition guarantees invertibility?

(A) and (B) both force $a = b = d$, giving $M = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ with $\det = 0$ – always singular, false. (C): a diagonal matrix with nonzero entries has $\det = ad \neq 0$ – invertible, true. (D): $\det M = ad - b^2$; if ad is not a perfect square it cannot equal b^2 (a perfect square), so $\det M \neq 0$ – invertible, true.
Answer: (C), (D)

Q.11

a, b, c positive integers, b/a an integer, a, b, c in GP, AM of a, b, c is $b + 2$. Find $\dfrac{a^2 + a - 14}{a + 1}$.

Write $b = ak$, $c = ak^2$ for integer ratio k. The AM condition $a + c = 2b + 6$ becomes $a(k - 1)^2 = 6$. Testing integer $k > 1$: only $k = 2$ gives an integer a (namely $a = 6$), since $(k - 1)^2$ must divide 6 and be a perfect square, forcing $(k - 1)^2 = 1$. So $a = 6$, and $\dfrac{36 + 6 - 14}{7} = \dfrac{28}{7} = 4$.
Answer: 4

Q.12

$n \geq 2$ points on a circle; adjacent-point segments coloured blue, rest red. Red count = blue count. Find n.

Total segments $= \binom{n}{2}$; blue (adjacent) segments $= n$. Setting $\binom{n}{2} - n = n$ gives $\dfrac{n(n - 1)}{2} = 2n \Rightarrow n - 1 = 4 \Rightarrow n = 5$.
Answer: 5

Q.13

$n_1 < n_2 < n_3 < n_4 < n_5$ positive integers summing to 20. Count the arrangements.

Setting $a_i = n_i - (i-1)$ converts the strictly increasing condition into $a_1 \le a_2 \le \cdots \le a_5$ with $\sum a_i = 20 - 10 = 10$ – the number of partitions of 10 into exactly 5 positive parts, equivalently partitions of 5 into at most 5 parts, i.e. all partitions of 5: 5; 4+1; 3+2; 3+1+1; 2+2+1; 2+1+1+1; 1+1+1+1+1 – a total of 7.

Answer: 7

Q.14

$f(x) = |x| + 1$, $g(x) = x^2 + 1$; $h(x) = \max\{f, g\}$ for $x \le 0$, $\min\{f, g\}$ for $x > 0$. Count points where h is not differentiable.

For $x \le 0$: $f \ge g$ exactly on $[-1, 0]$, so $h = g = x^2 + 1$ for $x < -1$ and $h = f = 1 - x$ for $-1 \le x \le 0$; the one-sided derivatives at $x = -1$ are -2 and -1 – not differentiable there. For $x > 0$: $f \le g$ on $(0, 1)$ (giving $h = g = x^2 + 1$ there) and $f \le g$ reverses for $x > 1$ (giving $h = f = x + 1$); derivatives at $x = 1$ are 2 and 1 – not differentiable. At $x = 0$: left derivative (from $1 - x$) is -1, right derivative (from $x^2 + 1$) is 0 – not differentiable. Points: $x = -1, 0, 1$.

Answer: 3

Q.15

$$\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2}(1 - x^2)^5 \right\} dx.$$

Integrating by parts (boundary terms vanish since $u'(0) = u'(1) = 0$ where $u = (1 - x^2)^5$): the integral equals $120 \int_0^1 x^3(1 - x^2)^4 dx$. Substituting $w = x^2$: $= 60 \int_0^1 w(1 - w)^4 dw = 60\, B(2, 5) = 60 \cdot \dfrac{1!\,4!}{6!} = 60 \cdot \dfrac{1}{30} = 2$.

Answer: 2

Q.16

Slope of the tangent to $(y - x^5)^2 = x(1 + x^2)^2$ at $(1, 3)$.

Implicit differentiation: $2(y - x^5)(y' - 5x^4) = (1 + x^2)^2 + 4x^2(1 + x^2) = (1 + x^2)(1 + 5x^2)$. At $(1, 3)$: $y - x^5 = 2$, so $4(y' - 5) = (2)(6) = 12 \Rightarrow y' - 5 = 3 \Rightarrow y' = 8$.

Answer: 8

Q.17

Largest nonnegative integer a such that $\displaystyle\lim_{x \to 1} \left\{ \frac{-ax + \sin(x - 1) + a}{x + \sin(x - 1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$.

With $h = x - 1 \to 0$, the base is $\dfrac{-ah + \sin h}{h + \sin h} \to \dfrac{(1 - a)h + O(h^3)}{2h + O(h^3)} \to \dfrac{1 - a}{2}$ for $a \ne 1$ (and $\to 0$ for $a = 1$), while the exponent $\dfrac{1 - x}{1 - \sqrt{x}} = 1 + \sqrt{x} \to 2$ exactly. For the base to be a well-defined positive real near $x = 1$ (required since the exponent is not an integer), we need $\dfrac{1-a}{2} > 0$, i.e. $a < 1$. Among nonnegative integers only $a = 0$ qualifies, giving limit $(1/2)^2 = 1/4$ as required; $a = 1$ gives limit 0, and $a \geq 2$ makes the base negative (undefined real power).

Answer: 0

Q.18

$f : [0, 4\pi] \to [0, \pi]$, $f(x) = \cos^{-1}(\cos x)$. Number of solutions of $f(x) = \dfrac{10-x}{10}$.

f is the triangular wave: x, $2\pi - x$, $x - 2\pi$, $4\pi - x$ on the successive intervals $[0, \pi]$, $[\pi, 2\pi]$, $[2\pi, 3\pi]$, $[3\pi, 4\pi]$. Solving against the line $1 - x/10$ on each branch gives $x = 10/11 \approx 0.91 \in [0, \pi]$ (valid), $x = (20\pi - 10)/9 \approx 5.87 \in [\pi, 2\pi]$ (valid), $x = (10 + 20\pi)/11 \approx 6.62 \in [2\pi, 3\pi]$ (valid), and $x = (40\pi - 10)/9 \approx 12.85 \notin [3\pi, 4\pi] \approx [9.42, 12.57]$ (invalid). So 3 solutions.

Answer: 3

Q.19

$d_1(P), d_2(P)$ = distances from $x - y = 0$, $x + y = 0$. Area of the first-quadrant region with $2 \leq d_1(P) + d_2(P) \leq 4$.

For (x, y) in the first quadrant, $d_1 + d_2 = \dfrac{|x-y| + (x+y)}{\sqrt{2}} = \sqrt{2}\max(x, y)$. The condition becomes $\sqrt{2} \leq \max(x, y) \leq 2\sqrt{2}$. This is the square of side $2\sqrt{2}$ (area 8) minus the square of side $\sqrt{2}$ (area 2): $8 - 2 = 6$.

Answer: 6

Q.20

$\vec{a}, \vec{b}, \vec{c}$ non-coplanar unit vectors, pairwise angle $\pi/3$. $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$. Find $(p^2 + 2q^2 + r^2)/q^2$.

Dotting with $\vec{a}, \vec{b}, \vec{c}$ (using $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ and cyclic invariance of the scalar triple product $V = [\vec{a}, \vec{b}, \vec{c}]$) gives three equations that solve to $p = r = -q$ and $V = p$. The Gram determinant of three unit vectors with all pairwise dot products $\frac{1}{2}$ is $\det\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{2}$, so $V^2 = \frac{1}{2}$, giving $p^2 = q^2 = r^2 = \frac{1}{2}$. Then $\dfrac{p^2 + 2q^2 + r^2}{q^2} = \dfrac{\frac{1}{2} + 1 + \frac{1}{2}}{\frac{1}{2}} = 4$.

Answer: 4