QuestIQ Academy

JEE Advanced 2015 – Paper 1 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single-digit integer; Section 2 = one or more correct; Section 3 = matching-type.

Q.1

Number of distinct solutions of $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ *in* $[0, 2\pi]$.

Using $\cos^4 x + \sin^4 x = 1 - \frac{1}{2}\sin^2 2x$ and $\cos^6 x + \sin^6 x = 1 - \frac{3}{4}\sin^2 2x$, the equation becomes $\frac{5}{4}\cos^2 2x + 2 - \frac{5}{4}\sin^2 2x = 2$, i.e. $\cos 4x = 0$. Over $[0, 2\pi]$, $4x$ ranges over $[0, 8\pi]$, giving 8 solutions of $\cos 4x = 0$.

Answer: 8

Q.2

C is the mirror image of $y^2 = 4x$ *in the line* $x + y + 4 = 0$*. If* A, B *are the points where* C *meets* $y = -5$*, find* AB*.*

Reflection in $x + y + 4 = 0$ sends $(x, y) \mapsto (-y - 4, -x - 4)$. A point $(t^2, 2t)$ on the parabola maps to $(X, Y) = (-2t - 4, -t^2 - 4)$. Eliminating $t = -\frac{X+4}{2}$ gives $Y = -\frac{(X+4)^2}{4} - 4$. Setting $Y = -5$: $(X + 4)^2 = 4 \Rightarrow X = -2, -6$, so $AB = 4$.

Answer: 4

Q.3

Minimum number of tosses of a fair coin so that $P(\text{at least 2 heads}) \geq 0.96$.

$P(\geq 2 \text{ heads}) = 1 - \dfrac{n+1}{2^n}$. For $n = 7$: $\dfrac{8}{128} = 0.0625 > 0.04$. For $n = 8$: $\dfrac{9}{256} \approx 0.0352 < 0.04$. So $n = 8$.

Answer: 8

Q.4

n *= ways 5 boys, 5 girls stand with all girls consecutive;* m *= ways with exactly 4 girls consecutive. Find* m/n*.*

$n = 6! \cdot 5! = 86400$ (girl-block as one unit). For m: choose which 4 of 5 girls form the block ($\binom{5}{4} = 5$), arrange them ($4! = 24$), then arrange this block, the 5 boys and the remaining girl (7 items) so that the lone girl is not adjacent to the block: $7! - 2 \cdot 6! = 3600$. So $m = 5 \cdot 24 \cdot 3600 = 432000$, and $m/n = 5$.
Answer: 5

Q.5

Normals to $y^2 = 4x$ at the ends of its latus rectum are tangent to $(x-3)^2 + (y+2)^2 = r^2$. Find r^2.

Latus rectum ends: $(1, 2)$ and $(1, -2)$. Normal at $(1, 2)$ ($t = 1$): $x + y = 3$. Normal at $(1, -2)$ ($t = -1$): $x - y = 3$. Distance from $(3, -2)$ to $x + y - 3 = 0$ is $\dfrac{|3-2-3|}{\sqrt{2}} = \sqrt{2}$, so $r^2 = 2$.
Answer: 2

Q.6

$f(x) = [x]$ for $x \le 2$, $f(x) = 0$ for $x > 2$ ($[\cdot]$ = floor). $I = \displaystyle\int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$. Find $4I - 1$.

On $(-1, 1)$, $x^2 < 1$ so $f(x^2) = 0$, contributing 0. On $[1, \sqrt{2})$, $x^2 \in [1, 2)$ so $f(x^2) = 1$, while $x+1 > 2$ gives $f(x+1) = 0$: contributes $\int_1^{\sqrt{2}} \frac{x}{2} dx = \frac{1}{4}$. On $[\sqrt{2}, 2]$, $x^2 \ge 2$ so $f(x^2) = 0$, contributing 0. So $I = \frac{1}{4}$ and $4I - 1 = 0$.
Answer: 0

Q.7

Cylindrical container, fixed inner volume V, wall thickness 2 mm, open top, solid base of thickness 2 mm and radius = outer radius. Material volume is minimised when the inner radius is 10 mm. Find $V/(250\pi)$.

With inner radius r, height $h = V/(\pi r^2)$, material volume $M(r) = 2\pi(r+2)^2 + \left[(r+2)^2 - r^2\right] \cdot \dfrac{V}{r^2} = 2\pi(r+2)^2 + \dfrac{4V(r+1)}{r^2}$. Differentiating, $M'(r) = 4(r+2)\left[\pi - V/r^3\right]$, so the minimum occurs at $V = \pi r^3$. With $r = 10$: $V = 1000\pi$, so $V/(250\pi) = 4$.
Answer: 4

Q.8

$F(x) = \displaystyle\int_x^{x^2+\pi/6} 2\cos^2 t \, dt$, $f : [0, \frac{1}{2}] \to [0, \infty)$ continuous. For $a \in [0, \frac{1}{2}]$, $F'(a) + 2$ equals the area bounded by $x = 0, y = 0, y = f(x), x = a$. Find $f(0)$.

By Leibniz's rule, $F'(x) = 4x\cos^2(x^2 + \pi/6) - 2\cos^2 x$. Since the area equals $\int_0^a f(x)dx = F'(a) + 2$, differentiating both sides gives $f(a) = F''(a)$. Computing $F''(x) = 4\cos^2(x^2 + \pi/6) - 16x^2\cos(x^2 + \pi/6)\sin(x^2 + \pi/6) + 4\sin x \cos x$, at $x = 0$ this is $4\cos^2(\pi/6) = 4 \cdot \frac{3}{4} = 3$.
Answer: 3

Q.9

X, Y non-zero 3×3 skew-symmetric matrices, Z non-zero symmetric. Which of $Y^3Z^4 - Z^4Y^3$, $X^{44} + Y^{44}$, $X^4Z^3 - Z^3X^4$, $X^{23} + Y^{23}$ is (are) skew-symmetric?

Odd powers of a skew-symmetric matrix are skew; even powers are symmetric; odd powers of a symmetric matrix are symmetric. Checking transposes: $(Y^3Z^4 - Z^4Y^3)^T = Y^3Z^4 - Z^4Y^3$ itself (symmetric, not skew) – false. $X^{44} + Y^{44}$ is a sum of symmetric matrices – symmetric, false. $(X^4Z^3 - Z^3X^4)^T = Z^3X^4 - X^4Z^3 = -(X^4Z^3 - Z^3X^4)$ – skew, true. $X^{23} + Y^{23}$ is a sum of skew matrices – skew, true.
Answer: (C), (D)

Q.10

Which α satisfy $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$*?*

Writing column j as $(i + j\alpha)^2 = i^2 + 2ij\alpha + j^2\alpha^2$, each column is $u + 2j\alpha\, v + j^2\alpha^2 w$ with $u = (1, 4, 9)^T, v = (1, 2, 3)^T, w = (1, 1, 1)^T$. Expanding the trilinear determinant over permutations of $\{u, v, w\}$ assigned to the three columns gives det $= 4\alpha^3 \det[u, v, w]$, and $\det[u, v, w] = -2$, so det $= -8\alpha^3$. Setting $-8\alpha^3 = -648\alpha$ gives $\alpha(\alpha^2 - 81) = 0$, so $\alpha = 0, \pm 9$ (excluding $\alpha = 0$ as trivial).
Answer: (B) 9, (C) -9

Q.11

$P_1 : y = 0$, $P_2 : x + z = 1$. $P_3 \neq P_1, P_2$ passes through $P_1 \cap P_2$, with distance 1 from $(0, 1, 0)$ and distance 2 from (α, β, γ). Which relations hold?

$P_3 : \lambda x + y + \lambda z - \lambda = 0$ for some λ. Distance from $(0, 1, 0)$: $\dfrac{|1 - \lambda|}{\sqrt{2\lambda^2 + 1}} = 1 \Rightarrow \lambda(\lambda + 2) = 0$; excluding $\lambda = 0$ (which gives P_1), $\lambda = -2$, so $P_3 : 2x - y + 2z - 2 = 0$. Distance from (α, β, γ): $\dfrac{|2\alpha - \beta + 2\gamma - 2|}{3} = 2 \Rightarrow 2\alpha - \beta + 2\gamma - 2 = \pm 6$, giving $2\alpha - \beta + 2\gamma - 8 = 0$ or $2\alpha - \beta + 2\gamma + 4 = 0$.
Answer: (B), (D)

Q.12

L is the line through the origin equidistant from $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$ (with each individual distance constant along L). M = locus of feet of perpendiculars from L to P_1. Which points lie on M?

Both planes have normal length $\sqrt{6}$, and equal distance at the origin forces L to lie in the bisector $3x + y = 0$; for the individual distances to stay constant, L's direction must be $\perp$ both normals: $n_1 \times n_2 = (1, -3, -5)$. So $L : (t, -3t, -5t)$. The foot of perpendicular to P_1 from this point has constant offset $\frac{1}{6}(1, 2, -1)$ (independent of t), giving $M : (t - \frac{1}{6}, -3t - \frac{1}{3}, -5t + \frac{1}{6})$. Testing $t = \frac{1}{6}$ gives $(0, -\frac{5}{6}, -\frac{2}{3})$; testing $t = 0$ gives $(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6})$ – both match.
Answer: (A), (B)

Q.13

P, Q distinct on $y^2 = 2x$; circle with diameter PQ passes through the vertex O; P in the first quadrant; area of $\triangle OPQ = 3\sqrt{2}$. Find P.

Parametrise $P = (p^2/2, p), Q = (q^2/2, q)$. $OP \perp OQ \Rightarrow pq = -4$. Area $= \frac{1}{4}|pq||p-q| = |p-q| = 3\sqrt{2}$. With $pq = -4$: $(p-q)^2 = p^2 + q^2 + 8 = 18 \Rightarrow p^2 + q^2 = 10$, so $(p+q)^2 = 2 \Rightarrow p + q = \pm\sqrt{2}$. Solving the resulting quadratics gives $\{p, q\} = \{2\sqrt{2}, -\sqrt{2}\}$ or $\{\sqrt{2}, -2\sqrt{2}\}$; taking the positive value (first quadrant) gives $P = (4, 2\sqrt{2})$ or $P = (1, \sqrt{2})$.
Answer: (A), (D)

Q.14

$(1 + e^x)y' + ye^x = 1$, $y(0) = 2$. *Which is (are) true?*

The equation is $\dfrac{d}{dx}[y(1 + e^x)] = 1$, so $y(1 + e^x) = x + C$; using $y(0) = 2$ gives $C = 4$, so $y = \dfrac{x+4}{1+e^x}$. Then $y(-4) = 0$ (true); $y(-2) = 2/(1 + e^{-2}) \neq 0$ (false). For a critical point, $y' = 0 \Leftrightarrow y = e^{-x} \Leftrightarrow e^{-x} = x + 3$; with $g(x) = e^{-x} - x - 3$, $g(-1) = e - 2 > 0$ and $g(0) = -2 < 0$, so by the IVT a critical point exists in $(-1, 0)$ (true), hence (D) is false.
Answer: (A), (C)

Q.15

Family of circles centred on $y = x$ satisfies $Py'' + Qy' + 1 = 0$. Which is (are) true?

Differentiating $(x - a)^2 + (y - a)^2 = r^2$ twice and eliminating a, r leads to $(x - y)y'' = (1 + y')(1 + (y')^2)$, i.e. $(y - x)y'' + \left[(y')^2 + y' + 1\right]y' + 1 = 0$. Matching to $Py'' + Qy' + 1 = 0$ gives $P = y - x$, $Q = 1 + y' + (y')^2$. Then $P + Q = 1 - x + y + y' + (y')^2$ (true), while $P - Q = -1 - x + y - y' - (y')^2 \neq x + y - y' - (y')^2$ (false).
Answer: (B), (C)

Q.16

$g : \mathbb{R} \to \mathbb{R}$ differentiable, $g(0) = 0, g'(0) = 0, g'(1) \neq 0$. $f(x) = \frac{x}{|x|}g(x)$ $(x \neq 0)$, $f(0) = 0$; $h(x) = e^{|x|}$. Check differentiability at $x = 0$ of $f, h, f \circ h, h \circ f$.

$f'(0) = \lim \operatorname{sgn}(x)g(x)/x = \pm g'(0) = 0$ from both sides – f is differentiable at 0 (true). $h'(0^+) = 1 \neq -1 = h'(0^-)$ – h is not differentiable (false). Since $e^{|x|} > 0$ always, $f \circ h(x) = g(e^{|x|})$; its one-sided derivatives at 0 are $\pm e\, g'(1)$, unequal since $g'(1) \neq 0$ – not differentiable (false). $h \circ f(x) = e^{|g(x)|}$; since $g(x) = o(x)$ (as $g'(0) = 0$), $\dfrac{e^{|g(x)|} - 1}{x} \approx \dfrac{|g(x)|}{x} \to 0$ – differentiable, with derivative 0 (true).
Answer: (A), (D)

Q.17

$f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$, $g(x) = \frac{\pi}{2}\sin x$. Which is (are) true regarding range of f, range of $f \circ g$, $\lim_{x\to 0} f(x)/g(x)$, and existence of x with $(g \circ f)(x) = 1$?

Since $\sin x \in [-1,1] \Rightarrow \frac{\pi}{2}\sin x \in [-\frac{\pi}{2},\frac{\pi}{2}] \Rightarrow \sin(\cdot) \in [-1,1] \Rightarrow \frac{\pi}{6}\sin(\cdot) \in [-\frac{\pi}{6},\frac{\pi}{6}] \Rightarrow f \in [-\frac{1}{2},\frac{1}{2}]$ (true); the identical chain of bounds applies to $f \circ g$, giving the same range $[-\frac{1}{2},\frac{1}{2}]$ (true). For small x: $f(x) \approx \frac{\pi}{6}\sin(\frac{\pi}{2}\sin x) \approx \frac{\pi}{6}\cdot\frac{\pi}{2}\sin x = \frac{\pi}{6}g(x)$, so the limit is $\frac{\pi}{6}$ (true). Since $f(x) \in [-\frac{1}{2},\frac{1}{2}]$, $g(f(x)) = \frac{\pi}{2}\sin(f(x)) \le \frac{\pi}{2}\sin\frac{1}{2} \approx 0.75 < 1$, so $(g \circ f)(x) = 1$ is never attained (false).

Answer: (A), (B), (C)

Q.18

$\triangle PQR$: $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}, \vec{c} = \overrightarrow{PQ}$. $|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}, \vec{b}\cdot\vec{c} = 24$. Which is (are) true?

Since $\vec{a}+\vec{b}+\vec{c} = 0$, $\vec{c} = -(\vec{a}+\vec{b})$, so $\vec{b}\cdot\vec{c} = -\vec{a}\cdot\vec{b} - |\vec{b}|^2 = 24 \Rightarrow \vec{a}\cdot\vec{b} = -72$ (matches (D)). Then $|\vec{c}|^2 = |\vec{a}+\vec{b}|^2 = 144 + 2(-72) + 48 = 48$, so $|\vec{c}|^2/2 - |\vec{a}| = 24 - 12 = 12$ (matches (A)); $|\vec{c}|^2/2 + |\vec{a}| = 36 \ne 30$ (false). Since $\vec{c}\times\vec{a} = -(\vec{a}+\vec{b})\times\vec{a} = -\vec{b}\times\vec{a} = \vec{a}\times\vec{b}$, we get $\vec{a}\times\vec{b} + \vec{c}\times\vec{a} = 2\vec{a}\times\vec{b}$; with $\cos\theta = \vec{a}\cdot\vec{b}/(|\vec{a}||\vec{b}|) = -\sqrt{3}/2$, $\sin\theta = 1/2$, so $|\vec{a}\times\vec{b}| = 24\sqrt{3}$, giving $|2\vec{a}\times\vec{b}| = 48\sqrt{3}$ (matches (C)).

Answer: (A), (C), (D)

Q.19

Match Column I to Column II.
(A) In $\mathbb{R}^2$, projection of $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ has magnitude $\sqrt{3}$; $\alpha = 2 + \sqrt{3}\beta$. Possible $|\alpha|$?
(B) $f(x) = -3ax^2 - 2\ (x < 1)$, $bx + a^2\ (x \ge 1)$ differentiable everywhere. Possible a?
(C) $\omega \ne 1$ cube root of unity; $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$. Possible n?
(D) HM of positive reals a, b is 4; $a, 5, q, b$ in AP, $q > 0$. Possible $|q - a|$?

(A) $|\sqrt{3}\alpha + \beta| = 2\sqrt{3}$; substituting $\alpha = 2 + \sqrt{3}\beta$ gives $|2\sqrt{3} + 4\beta| = 2\sqrt{3}$, so $\beta = 0$ ($\alpha = 2$) or $\beta = -\sqrt{3}$ ($\alpha = -1$): $|\alpha| \in \{1, 2\}$. (B) Continuity and differentiability at $x = 1$ give $b = -6a$ and $b = -(a+1)(a+2)$, so $a^2 - 3a + 2 = 0 \Rightarrow a = 1, 2$. (C) Writing $y = x\omega, z = x\omega^2$ (verified directly using $\omega^3 = 1$), the sum is $x^{4n+3}(1 + \omega^n + \omega^{2n})$, which vanishes iff $n \not\equiv 0 \pmod 3$; among $\{1,2,3,4,5\}$ this gives $n \in \{1,2,4,5\}$. (D) $2ab/(a+b) = 4 \Rightarrow ab = 2(a+b)$; with $d = 5 - a$, $q = 10 - a$, $b = 15 - 2a$, solving $2a^2 - 17a + 30 = 0$ gives $a = 6$ ($q = 4, |q-a| = 2$) or $a = 2.5$ ($q = 7.5, |q-a| = 5$).

Answer: (A)→P,Q; (B)→P,Q; (C)→P,Q,S,T; (D)→Q,T

Q.20

Match Column I to Column II.
(A) $\triangle XYZ$: $2(a^2 - b^2) = c^2$, $\lambda = \sin(X - Y)/\sin Z$. Possible n with $\cos(n\pi\lambda) = 0$?
(B) $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$. Possible a/b?
(C) X, Y at $(\sqrt{3}, 1), (1, \sqrt{3})$; $Z = (\beta, 1-\beta)$; distance of Z from the acute-angle bisector of $\overrightarrow{OX}, \overrightarrow{OY}$ is $3/\sqrt{2}$. Possible $|\beta|$?

(D) $F(\alpha)$ = area bounded by $x = 0, x = 2, y^2 = 4x, y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$. Find $F(\alpha) + \frac{8}{3}\sqrt{2}$ for $\alpha = 0, 1$.

(A) Using the identity $\sin(X - Y)/\sin Z = (a^2 - b^2)/c^2 = \frac{1}{2}$, so $\lambda = \frac{1}{2}$; $\cos(n\pi/2) = 0$ for odd n: among $\{1, 2, 3, 5, 6\}$ this gives $n \in \{1, 3, 5\}$. (B) Simplifying with double-angle identities gives $2\sin^2 Y - \sin X \sin Y - \sin^2 X = 0$, a quadratic in $\sin Y$ giving $\sin Y = \sin X$ (the other root is negative), so $X = Y$ and $a/b = 1$. (C) The acute bisector is the line $y = x$; distance $\dfrac{|2\beta - 1|}{\sqrt{2}} = \dfrac{3}{\sqrt{2}} \Rightarrow \beta = 2$ or $\beta = -1$, so $|\beta| \in \{1, 2\}$. (D) For $\alpha = 0$: $y = 3$ (constant), area $= \int_0^2 (3 - 2\sqrt{x})dx = 6 - \frac{8\sqrt{2}}{3}$, so $F(0) + \frac{8\sqrt{2}}{3} = 6$. For $\alpha = 1$: $y = 3 - x$ on $[0, 1]$, $y = 1 + x$ on $[1, 2]$ (touching the parabola at $x = 1$), giving area $5 - \frac{8\sqrt{2}}{3}$, so $F(1) + \frac{8\sqrt{2}}{3} = 5$.

Answer: (A)→P,R,S; (B)→P; (C)→P,Q; (D)→S,T