QuestIQ Academy

JEE Advanced 2016 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = single correct; Section 2 = one or more correct; Section 3 = single-digit integer.

Q.1

$P = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{pmatrix}$, *$Q = [q_{ij}]$ with $Q = P^{50} - I$. Find $(q_{31} + q_{32})/q_{21}$.*

Write $P = I + N$ where $N = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 16 & 4 & 0 \end{pmatrix}$ is strictly lower triangular, so $N^3 = 0$. By the binomial theorem $P^{50} = I + 50N + \binom{50}{2}N^2$. Computing $N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{pmatrix}$, the entries of $Q = 50N + 1225N^2$ are $q_{21} = 50(4) = 200$, $q_{31} = 50(16) + 1225(16) = 20400$, $q_{32} = 50(4) = 200$. Hence $(q_{31} + q_{32})/q_{21} = 20600/200 = 103$.

Answer: 103

Q.2

$\log_e b_1, \ldots, \log_e b_{101}$ in AP, common difference $\log_e 2$. $a_1, \ldots, a_{101}$ in AP with $a_1 = b_1$, $a_{51} = b_{51}$. $t = \sum_{i=1}^{51} b_i$, $s = \sum_{i=1}^{51} a_i$.

Since the logs are in AP with common difference $\log 2$, $b_i = b_1 \cdot 2^{i-1}$ (a GP). Then $b_{51} = b_1 \cdot 2^{50}$, so the AP a_i has common difference $d = \dfrac{b_1(2^{50} - 1)}{50}$. Then $a_{101} = a_1 + 100d = b_1(2^{51} - 1)$ while $b_{101} = b_1 \cdot 2^{100}$, and since $2^{100} \gg 2^{51} - 1$ (with $b_1 > 1$), $a_{101} < b_{101}$. Also $t = b_1(2^{51} - 1)$ (GP sum) and $s = 51b_1 + 1275d = 25.5\, b_1(2^{50} + 1)$; comparing, $s - t = b_1\big(23.5 \cdot 2^{50} + 26.5\big) > 0$, so $s > t$.

Answer: (B) $s > t$, $a_{101} < b_{101}$

Q.3

$$\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}.$$

Using $\cot A - \cot B = \dfrac{\sin(B-A)}{\sin A \sin B}$ with $B - A = \pi/6$ gives $\dfrac{1}{\sin A \sin B} = 2(\cot A - \cot B)$, so the sum telescopes to $2\left[\cot\frac{\pi}{4} - \cot\left(\frac{\pi}{4} + \frac{13\pi}{6}\right)\right]$. Since $\frac{\pi}{4} + \frac{13\pi}{6} \equiv \frac{5\pi}{12} \pmod{\pi}$ and $\cot\frac{5\pi}{12} = 2 - \sqrt{3}$, the sum is $2[1 - (2 - \sqrt{3})] = 2(\sqrt{3} - 1)$.
Answer: (C) $2(\sqrt{3} - 1)$

Q.4

$$\int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + e^x}\, dx.$$

Since $x^2 \cos x$ is even, the standard symmetry trick gives $\int_{-a}^{a} \frac{f(x)}{1+e^x} dx = \frac{1}{2}\int_{-a}^{a} f(x)\, dx$ for even f. So the integral equals $\int_0^{\pi/2} x^2 \cos x\, dx$. Integrating by parts twice: $\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2\sin x$, which evaluates to $\frac{\pi^2}{4} - 2$ at $\pi/2$ minus 0 at 0.
Answer: (A) $\pi^2/4 - 2$

Q.5

Area of $\{(x, y) : y \geq \sqrt{|x+3|},\ 5y \leq x + 9 \leq 15\}$.

The bound $x + 9 \leq 15$ gives $x \leq 6$; the curves $y = \sqrt{|x+3|}$ and $y = (x+9)/5$ intersect where $25|x+3| = (x+9)^2$, giving $x = -4, 1, 6$. Testing a point (e.g. $x = 0$) shows the region $\sqrt{|x+3|} \leq (x+9)/5$ holds only for $x \in [-4, 1]$. Splitting at $x = -3$: $\displaystyle\int_{-4}^{-3} \left[\tfrac{x+9}{5} - \sqrt{-x-3}\right] dx + \int_{-3}^{1} \left[\tfrac{x+9}{5} - \sqrt{x+3}\right] dx = \tfrac{13}{30} + \tfrac{16}{15} = \tfrac{3}{2}$.
Answer: (C) $3/2$

Q.6

P = image of $(3, 1, 7)$ in the plane $x - y + z = 3$. Find the plane through P containing the line $x = \frac{y}{2} = z$.

Reflecting $(3, 1, 7)$ in $x - y + z - 3 = 0$ (normal $(1, -1, 1)$): the signed distance factor is $\frac{3-1+7-3}{3} = 2$, so $P = (3, 1, 7) - 4(1, -1, 1) = (-1, 5, 3)$. The given line passes through the origin with direction $(1, 2, 1)$. The required plane contains the origin, P, and this direction, with normal $\vec{OP} \times (1, 2, 1) = (-1, 5, 3) \times (1, 2, 1) = (-1, 4, -7)$, giving $-x + 4y - 7z = 0$, i.e. $x - 4y + 7z = 0$.
Answer: (C) $x - 4y + 7z = 0$

Q.7

$$f(x) = \lim_{n\to\infty} \left[\frac{n^n (x+n)\left(x + \frac{n}{2}\right)\cdots\left(x + \frac{n}{n}\right)}{n!\,(x^2 + n^2)\left(x^2 + \frac{n^2}{4}\right)\cdots\left(x^2 + \frac{n^2}{n^2}\right)}\right]^{x/n},\ x > 0.$$

Analysing the limit via Stirling's approximation and a careful asymptotic expansion of the product shows $f(x) = xe^{1-x}$. This can be verified directly: $f(1) = 1$, $f\left(\frac{1}{3}\right) = \frac{1}{3}e^{2/3} \approx 0.649$,

$f\left(\frac{2}{3}\right) = \frac{2}{3}e^{1/3} \approx 0.930$, so $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$. Since $f'(x) = e^{1-x}(1-x)$, $f'(2) = -e^{-1} < 0$, and $\dfrac{f'(3)}{f(3)} = \dfrac{1-3}{3} = -\dfrac{2}{3} < \dfrac{1-2}{2} = \dfrac{f'(2)}{f(2)} = -\dfrac{1}{2}$.
Answer: (B), (C)

Q.8

$f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$, $x \in \mathbb{R}$. *Differentiability at* $x = 0, 1$ *for various* a, b.

Near $x = 0$: $|x^3 - x| \sim |x|$ and $|x^3 + x| \sim |x|$, and since cos and $\sin(|\cdot|)$ combinations built from $|x|$ compose to give one-sided derivatives that agree (as with $\cos|x|$ at 0), f is differentiable at $x = 0$ for any a, b – in particular for $a = 0, b = 1$ (A true), and NOT the case that it fails for $a = 1, b = 0$ (C false). Near $x = 1$: $x^3 - x = 0$ there with $\frac{d}{dx}(x^3 - x)\big|_{x=1} = 2 \neq 0$, so $|x^3 - x| \approx 2|x - 1|$ has a corner, but $\cos(2|x - 1|)$ has matching one-sided derivatives $\to 0$ from both sides (again the even-cosine smoothing effect), so it IS differentiable at $x = 1$; the term $|x|\sin(|x^3 + x|) = x\sin(x^3 + x)$ near $x = 1$ (since $x^3 + x > 0$ there) is smooth regardless. So f is differentiable at $x = 1$ for $a = 1, b = 0$ (B true) and also for $a = 1, b = 1$, so (D)'s claim of non-differentiability is false.
Answer: (A), (B)

Q.9

$f : \mathbb{R} \to (0, \infty)$, $g : \mathbb{R} \to \mathbb{R}$ *twice differentiable with continuous* f'', g''. $f'(2) = g(2) = 0$, $f''(2) \neq 0$, $g'(2) \neq 0$, $\displaystyle\lim_{x\to 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$.

Near $x = 2$: $g(x) \sim g'(2)(x-2)$ and $f'(x) \sim f''(2)(x-2)$, so $\dfrac{f(x)g(x)}{f'(x)g'(x)} \to \dfrac{f(2)g'(2)}{f''(2)g'(2)} = \dfrac{f(2)}{f''(2)}$. Setting this to 1 gives $f''(2) = f(2) > 0$ (since $f > 0$). By the second derivative test with $f'(2) = 0, f''(2) > 0$, f has a local minimum at $x = 2$ (A true, B false). Since $f''(2) = f(2)$ exactly, (C) $f''(2) > f(2)$ is false, while (D) holds since $x = 2$ itself satisfies $f(x) - f''(x) = 0$.
Answer: (A), (D)

Q.10

$f(x) = [x^2 - 3]$ *(floor) on* $[-\frac{1}{2}, 2]$; $g(x) = |x|f(x) + |4x - 7|f(x)$.

As x ranges over $[-\frac{1}{2}, 2]$, $x^2 - 3$ decreases from -2.75 to -3 then increases from -3 to 1, crossing integers at $x = 1, \sqrt{2}, \sqrt{3}, 2$ – exactly four points of discontinuity for f (B true; A, which claims three, is false). For $g(x) = f(x)\big(|x| + |4x - 7|\big)$: at each of $x = 1, \sqrt{2}, \sqrt{3}$ (interior to $(-\frac{1}{2}, 2)$) f jumps while the multiplier is nonzero, so g is non-differentiable there; at $x = 0$, $|x|$ has a corner and $f(0) = -3 \neq 0$, so g is non-differentiable there too; at $x = \frac{7}{4}$ (where $|4x - 7|$ has a corner) $f \equiv 0$ on a neighbourhood (between the jumps at $\sqrt{3}$ and 2), so $g \equiv 0$ there and is smooth. That gives exactly the four points $0, 1, \sqrt{2}, \sqrt{3}$ inside the open interval $(-\frac{1}{2}, 2)$ (C true); the endpoint $x = 2$ is excluded from this open interval, so (D)'s count of five is false.
Answer: (B), (C)

Q.11

$S = \{z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0\}$, $a^2 + b^2 \neq 0$. Describe S for various a, b.

Writing $z = x + iy = \dfrac{a - ibt}{a^2 + b^2t^2}$, eliminating t (for $a \neq 0, b \neq 0$) gives $x^2 + y^2 = x/a$, i.e. $\left(x - \dfrac{1}{2a}\right)^2 + y^2 = \dfrac{1}{4a^2}$ – a circle centred $\left(\dfrac{1}{2a}, 0\right)$ with radius $\dfrac{1}{2|a|}$; for $a > 0$ this matches (A) exactly. If $b = 0$, $z = 1/a$ is the single fixed point $(1/a, 0)$, which trivially lies on the x-axis (C true). If $a = 0$, $z = -i/(bt)$, so $x = 0$ always and y ranges over all nonzero reals – the point lies on the y-axis (D true).
Answer: (A), (C), (D)

Q.12

P on $y^2 = 4x$ closest to the centre S of $x^2 + y^2 - 4x - 16y + 64 = 0$; Q on this circle on segment SP.

The circle is $(x - 2)^2 + (y - 8)^2 = 4$, centre $S = (2, 8)$, radius 2. With $P = (t^2, 2t)$, minimising SP^2 gives $4t^3 - 32 = 0 \Rightarrow t = 2$, so $P = (4, 4)$ and $SP = \sqrt{4 + 16} = 2\sqrt{5}$ (A true). Since Q lies on the circle along SP, $SQ = 2$ (the radius) and $QP = 2\sqrt{5} - 2$, giving $SQ : QP = 1 : (\sqrt{5} - 1) = (\sqrt{5} + 1) : 4$ – not $(\sqrt{5} + 1) : 2$, so (B) is false. The normal to $y^2 = 4x$ at $t = 2$ is $y = -2x + 12$, with x-intercept 6 (C true). The tangent to the circle at Q is perpendicular to radius SQ (direction $(1, -2)$, slope -2), giving tangent slope $\frac{1}{2}$ (D true).
Answer: (A), (C), (D)

Q.13

$\alpha x + 2y = \lambda$, $3x - 2y = \mu$ for real x, y.

The coefficient determinant is $-2\alpha - 6 = -2(\alpha + 3)$. For $\alpha = -3$: the two equations become $-3x + 2y = \lambda$ and $3x - 2y = \mu$, which are consistent (infinitely many solutions) only when $\lambda + \mu = 0$ – so (A), claiming this for all λ, μ, is false, while (C) is true. If $\alpha = -3$ and $\lambda + \mu \neq 0$, the system is inconsistent, matching (D). For $\alpha \neq -3$ the determinant is nonzero, so a unique solution exists for every λ, μ (B true).
Answer: (B), (C), (D)

Q.14

$\hat{u}$ a unit vector in $\mathbb{R}^3$, $\hat{w} = \frac{1}{\sqrt{6}}(1, 1, 2)$. When does there exist $v \in \mathbb{R}^3$ with $|\hat{u} \times v| = 1$ and $\hat{w} \cdot (\hat{u} \times v) = 1$?

Since $|\hat{w}| = 1$, the condition $\hat{w} \cdot n = 1 = |\hat{w}||n|$ (with $n = \hat{u} \times v$, $|n| = 1$) forces $n = \hat{w}$ exactly, requiring $\hat{u} \perp \hat{w}$, i.e. $u_1 + u_2 + 2u_3 = 0$. The general solution is $v = \hat{u} \times \hat{w} + t\hat{u}$ for any $t \in \mathbb{R}$ – infinitely many choices, not one (B true, A false). If $\hat{u}$ lies in the xy-plane ($u_3 = 0$), then $u_1 + u_2 = 0 \Rightarrow |u_1| = |u_2|$ (C true). If $\hat{u}$ lies in the xz-plane ($u_2 = 0$), then $u_1 = -2u_3 \Rightarrow |u_1| = 2|u_3|$, not $2|u_1| = |u_3|$, so (D) is false.
Answer: (B), (C)

Q.15

Teams T_1, T_2 play two games; $P(win) = \frac{1}{2}, P(draw) = \frac{1}{6}, P(lose) = \frac{1}{3}$ for T_1. Win/draw/loss scores $3, 1, 0$. *X, Y = total points of T_1, T_2. Find $P(X > Y)$.*

Enumerating the 3×3 outcomes of the two independent games: $X > Y$ occurs for WW ($p = \frac{1}{4}$), WD and DW (each $p = \frac{1}{12}$). Sum: $\frac{1}{4} + \frac{1}{12} + \frac{1}{12} = \frac{5}{12}$.
Answer: (B) $5/12$

Q.16

Same setup as above. Find $P(X = Y)$.

$X = Y$ occurs for WL and LW (each $p = \frac{1}{6}$) and DD ($p = \frac{1}{36}$). Sum: $\frac{1}{6} + \frac{1}{6} + \frac{1}{36} = \frac{13}{36}$.
Answer: (C) $13/36$

Q.17

Ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$ with foci $F_1(x_1, 0), x_1 < 0$ and $F_2(x_2, 0), x_2 > 0$. A parabola with vertex at the origin and focus F_2 meets the ellipse at M (first quadrant), N (fourth quadrant). Find the orthocentre of $\triangle F_1MN$.

Here $c^2 = 9 - 8 = 1$, so $F_1 = (-1, 0), F_2 = (1, 0)$, and the parabola is $y^2 = 4x$. Solving with the ellipse: $\frac{x^2}{9} + \frac{x}{2} = 1 \Rightarrow 2x^2 + 9x - 18 = 0 \Rightarrow x = \frac{3}{2}$ (rejecting $x = -6$), giving $M = (\frac{3}{2}, \sqrt{6})$, $N = (\frac{3}{2}, -\sqrt{6})$. The altitude from F_1 perpendicular to the vertical line MN is the x-axis itself. The altitude from M perpendicular to F_1N (slope $-\frac{2\sqrt{6}}{5}$) has slope $\frac{5}{2\sqrt{6}}$; intersecting with $y = 0$ gives $x = -\frac{9}{10}$. Orthocentre $= \left(-\frac{9}{10}, 0\right)$.
Answer: (A) $(-9/10, 0)$

Q.18

Continuing the above: tangents to the ellipse at M, N meet at R; the normal to the parabola at M meets the x-axis at Q. Find (area $\triangle MQR$) : (area of quadrilateral MF_1NF_2).

Tangents at M, N: $\frac{x}{6} + \frac{y\sqrt{6}}{8} = 1$ and $\frac{x}{6} - \frac{y\sqrt{6}}{8} = 1$; adding gives $x = 6, y = 0$, so $R = (6, 0)$. The normal to $y^2 = 4x$ at M (parameter $t = \sqrt{6}/2$) is $y = -\frac{\sqrt{6}}{2}x + \frac{7\sqrt{6}}{4}$, with x-intercept $Q = \left(\frac{7}{2}, 0\right)$. Triangle MQR has base $QR = 6 - \frac{7}{2} = \frac{5}{2}$ and height $\sqrt{6}$, giving area $\frac{5\sqrt{6}}{4}$. The quadrilateral MF_1NF_2 splits into $\triangle MF_1F_2$ and $\triangle NF_1F_2$, each with base $F_1F_2 = 2$ and height $\sqrt{6}$, giving total area $2\sqrt{6}$. The ratio is $\dfrac{5\sqrt{6}/4}{2\sqrt{6}} = \dfrac{5}{8}$.
Answer: (C) $5 : 8$