QuestIQ Academy

JEE Advanced 2019 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = one or more correct; Section 2 = numerical value; Section 3 = list-match.

Q.1

$P_1, \ldots, P_6$ are the six 3×3 permutation matrices; $X = \sum_{k=1}^{6} P_k A P_k^T$ with $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

Summing $P_\sigma A P_\sigma^T$ over all $\sigma \in S_3$: each diagonal entry becomes $2(\mathrm{tr} A) = 2(3) = 6$, and each off-diagonal entry becomes the sum of all off-diagonal entries of A, which is 12. So $X = \begin{pmatrix} 6 & 12 & 12 \\ 12 & 6 & 12 \\ 12 & 12 & 6 \end{pmatrix}$, symmetric (B true), with diagonal sum 18 (C true), and $X(1,1,1)^T = 30(1,1,1)^T$ so $\alpha = 30$ (A true). Since $(1,1,1)$ is an eigenvector of eigenvalue 30, $X - 30I$ is singular (D false).

Answer: (A), (B), (C)

Q.2

$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{pmatrix}$, $R = PQP^{-1}$.

Direct computation of the $(1,2)$ entries of PQ and QP gives $2x + 4$ and $2 + 2x$, which can never be equal, so no x makes $PQ = QP$ (A false). Since R is similar to Q, $\det R = \det Q = 48 - 4x^2$ for all x, matching the stated expression plus 8 (B true). At $x = 0$, Q is diagonal with eigenvalue 6 for eigenvector e_3; the corresponding eigenvector of R is $Pe_3 = (1,2,3)^T$, giving $a = 2, b = 3, a + b = 5$ (C true). At $x = 1$, $\det R = 44 \neq 0$, so R has no non-trivial null vector (D false).

Answer: (B), (C)

Q.3

$$f(n) = \frac{\sum_{k=0}^{n} \sin \frac{(k+1)\pi}{n+2} \sin \frac{(k+2)\pi}{n+2}}{\sum_{k=0}^{n} \sin^2 \frac{(k+1)\pi}{n+2}}.$$

Writing $N = n+2$ and using product-to-sum formulas along with $\sum_{j=1}^{N-1} \cos \frac{2j\pi}{N} = -1$ and $\sum_{j=0}^{N-1} \cos \frac{(2j+1)\pi}{N} = 0$, both sums simplify and $f(n) = \cos \frac{\pi}{n+2}$. (A) $f(4) = \cos(\pi/6) = \sqrt{3}/2$, true. (B) $\lim f(n) = \cos 0 = 1 \neq \frac{1}{2}$, false. (C) $f(6) = \cos(\pi/8)$, so $\alpha = \tan(\pi/8) = \sqrt{2}-1$, which satisfies $\alpha^2 + 2\alpha - 1 = 0$, true. (D) $f(5) = \cos(\pi/7)$, so $\sin(7\cos^{-1} f(5)) = \sin \pi = 0$, true.
Answer: (A), (C), (D)

Q.4

PROPERTY 1: $\lim_{h\to 0} \frac{f(h)-f(0)}{\sqrt{|h|}}$ *exists finite. PROPERTY 2:* $\lim_{h\to 0} \frac{f(h)-f(0)}{h^2}$ *exists finite.*

(A) $f(x) = |x|$: ratio $= \sqrt{|h|} \to 0$, PROPERTY 1 holds, true. (B) $f(x) = x^{2/3}$: ratio $= |h|^{1/6} \to 0$, PROPERTY 1 holds, true. (C) $f(x) = x|x|$: ratio $\to +1$ from the right and -1 from the left, limit fails to exist, false. (D) $f(x) = \sin x$: ratio $\sim 1/h \to \infty$, false.
Answer: (A), (B)

Q.5

$f(x) = \sin(\pi x)/x^2$, $x > 0$*;* $x_1 < x_2 < \cdots$ *local maxima,* $y_1 < y_2 < \cdots$ *local minima.*

Setting $f' = 0$ reduces to $\tan(\pi x) = \pi x/2$. Analysis interval-by-interval shows each maximum x_n lies in $(2n, 2n+\frac{1}{2})$ (C true) and each minimum y_n lies in $(2n-1, 2n-\frac{1}{2})$, so $y_1 < x_1$, making (A) false. An asymptotic expansion of the root of $\tan(\pi t) = \pi x/2$ near $t = \frac{1}{2}$ shows the gaps satisfy $x_{n+1} - x_n > 2$ (B true) and $x_n - y_n > 1$ (D true) for every n.
Answer: (B), (C), (D)

Q.6

$$\lim_{n\to\infty} \frac{1 + 2^{1/3} + \cdots + n^{1/3}}{n^{7/3}\left(\frac{1}{(an+1)^2} + \cdots + \frac{1}{(an+n)^2}\right)} = 54, \ |a| > 1.$$

The numerator $\sim \frac{3}{4}n^{4/3}$. The denominator sum is a Riemann sum: $\sum_{j=1}^{n} \frac{1}{(an+j)^2} \sim \frac{1}{n}\int_0^1 \frac{dx}{(a+x)^2} = \frac{1}{n\,a(a+1)}$, so the denominator $\sim n^{4/3}/(a(a+1))$. The limit becomes $\frac{3}{4}a(a+1) = 54 \Rightarrow a(a+1) = 72 \Rightarrow a = 8$ or $a = -9$.
Answer: (A), (D)

Q.7

$f(x) = (x-1)(x-2)(x-5)$, $F(x) = \int_0^x f(t)\,dt$, $x > 0$.

$F' = f$ changes sign $-, +, -, +$ at $x = 1, 2, 5$, giving a local minimum at $x = 1$ (A true), a local maximum at $x = 2$ (B true), and another local minimum at $x = 5$ – that is one maximum and two minima, so (C) is false. Direct integration gives $F(x) = \frac{x^4}{4} - \frac{8x^3}{3} + \frac{17x^2}{2} - 10x$, and $F(2) = -\frac{10}{3} < 0$, the largest value F attains on $(0, 5]$, so $F(x) \neq 0$ throughout $(0, 5)$ (D true).
Answer: (A), (B), (D)

Q.8

$L_1 : \vec{r} = \lambda\hat{\imath}$, $L_2 : \vec{r} = \hat{k} + \mu\hat{\jmath}$, $L_3 : \vec{r} = \hat{\imath} + \hat{\jmath} + \nu\hat{k}$. For which Q on L_2 do collinear $P \in L_1, R \in L_3$ exist?

Writing $Q = (1-t)P + tR$ and matching coordinates forces $t = q$ (the y-parameter of Q), with P's parameter $= q/(q-1)$ and R's parameter $= 1/q$ – both well-defined precisely when $q \neq 0, 1$. So $Q = \hat{k} - \frac{1}{2}\hat{\jmath}$ and $Q = \hat{k} + \frac{1}{2}\hat{\jmath}$ work; $Q = \hat{k}$ ($q = 0$) and $Q = \hat{k} + \hat{\jmath}$ ($q = 1$) do not.
Answer: (A), (C)

Q.9

$\det\begin{pmatrix} \sum k & \sum \binom{n}{k}k^2 \\ \sum \binom{n}{k}k & \sum \binom{n}{k}3^k \end{pmatrix} = 0$. *Find* $\sum_{k=0}^{n} \frac{\binom{n}{k}}{k+1}$.

Using $\sum k = \frac{n(n+1)}{2}$, $\sum \binom{n}{k}k = n2^{n-1}$, $\sum \binom{n}{k}k^2 = n(n+1)2^{n-2}$, $\sum \binom{n}{k}3^k = 4^n$, the determinant simplifies to $n(n+1)2^{2n-3}(4-n) = 0$, giving $n = 4$. Then $\sum_{k=0}^{4} \frac{\binom{4}{k}}{k+1} = \frac{2^5-1}{5} = \frac{31}{5}$.
Answer: 6.20

Q.10

Five persons in a circle, hats of 3 colours, adjacent persons differ. Number of valid colourings.

Proper colourings of a cycle C_n with k colours number $(k-1)^n + (-1)^n(k-1)$. For $n = 5, k = 3$: $2^5 - 2 = 30$.
Answer: 30.00

Q.11

$S = \{1, \ldots, 6\}$; independent events A, B; count ordered pairs (A, B) with $1 \leq |B| < |A|$.

Independence requires $|A \cap B| = |A||B|/6$. Checking all (a, b) with $b < a \leq 6$ for which $6 \mid ab$ gives valid triples (a, b, c): $(3, 2, 1), (4, 3, 2), (6, 1, 1), (6, 2, 2), (6, 3, 3), (6, 4, 4), (6, 5, 5)$. Counting subset choices $\binom{6}{c}\binom{6-c}{a-c}\binom{6-a}{b-c}$ for each and summing: $180 + 180 + 6 + 15 + 20 + 15 + 6 = 422$.
Answer: 422.00

Q.12

$\sec^{-1}\left[\frac{1}{4}\sum_{k=0}^{10} \sec\left(\frac{7\pi}{12} + \frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12} + \frac{(k+1)\pi}{2}\right)\right]$ *in* $[-\pi/4, 3\pi/4]$.

With $\theta_k = \frac{7\pi}{12} + \frac{k\pi}{2}$ (so $\theta_{k+1} - \theta_k = \pi/2$), the identity $\tan\theta_{k+1} - \tan\theta_k = \sec\theta_k \sec\theta_{k+1}$ telescopes the sum to $\tan\theta_{11} - \tan\theta_0 = (2 - \sqrt{3}) - (-(2 + \sqrt{3})) = 4$. So the bracket is 1, and $\sec^{-1}(1) = 0$.
Answer: 0.00

Q.13

$$\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\cos\theta} + \sqrt{\sin\theta})^5}\, d\theta.$$

Substituting $t = \sqrt{\tan\theta}$ gives $\sqrt{\cos\theta} = (1+t^4)^{-1/4}$ and $d\theta = \frac{2t}{1+t^4}dt$; remarkably the $(1+t^4)$ factors cancel and the integral (after using the $\theta \to \pi/2 - \theta$ symmetry to double it) reduces to $\frac{3}{2}\int_0^\infty \frac{2t}{(1+t)^4}dt = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}$.
Answer: 0.50

Q.14

$\vec{a} = 2\hat{\imath}+\hat{\jmath}-\hat{k}$, $\vec{b} = \hat{\imath}+2\hat{\jmath}+\hat{k}$, $\vec{c} = \alpha\vec{a}+\beta\vec{b}$. Projection of $\vec{c}$ on $\vec{a}+\vec{b}$ is $3\sqrt{2}$. Minimize $(\vec{c}-(\vec{a}\times\vec{b}))\cdot\vec{c}$.

$\vec{a}+\vec{b} = (3,3,0)$, $|\vec{a}+\vec{b}| = 3\sqrt{2}$, so $\vec{c}\cdot(\vec{a}+\vec{b}) = 18$, giving $\alpha+\beta = 2$. Since $\vec{a}\times\vec{b}$ is perpendicular to $\vec{c}$, the expression reduces to $\vec{c}\cdot\vec{c} = 6(\alpha^2+\alpha\beta+\beta^2)$. Substituting $\beta = 2-\alpha$ gives $6(\alpha^2-2\alpha+4)$, minimized at $\alpha = 1$, value $6(3) = 18$.
Answer: 18.00

Q.15

$f(x) = \sin(\pi\cos x)$, $g(x) = \cos(2\pi\sin x)$. $X = \{f = 0\}$, $Y = \{f' = 0\}$, $Z = \{g = 0\}$, $W = \{g' = 0\}$.

$X = \{k\pi/2\}$ is an AP; $Y = \{k\pi/3\}$ is also an AP and contains $\pi/3, 2\pi/3, \pi$.
Answer: (II), (Q), (T)

Q.16

Continuing Q.15's setup: match Z and W.

Z (solving $\cos x = \pm\frac{1}{4}, \pm\frac{3}{4}$) has irregular spacing – not an AP. $W = \{g' = 0\}$ works out to $\{0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\}$ (mod 2π), an irregular (non-AP) pattern that nonetheless contains $\{\pi/2, 3\pi/2, 4\pi, 7\pi\}$ and $\{\pi/6, 7\pi/6, 13\pi/6\}$.
Answer: (IV), (P), (R), (S)

Q.17

Circles $C_1 : x^2+y^2 = 9$, $C_2 : (x-3)^2+(y-4)^2 = 16$ meet at X, Y. C_3 has centre collinear with C_1, C_2's centres, contains both, and is internally tangent to each. Line XY meets C_3 at Z, W.

Solving the tangency conditions gives C_3 centre $(h,k) = (\frac{9}{5}, \frac{12}{5})$, radius 6, so $2h+k = 6$. The radical axis XY is $3x+4y = 9$; its chord in C_1 has length $24/5$, and its chord in C_3 (using the distance from $(\frac{9}{5}, \frac{12}{5})$ to the line, $6/5$) has length $24\sqrt{6}/5$. The ratio $ZW/XY = \sqrt{6}$.
Answer: (II), (Q)

Q.18

Continuing Q.17's setup: the tangent to C_1 at the tangency point M is also tangent to the parabola $x^2 = 8\alpha y$.

$M = (-\frac{9}{5}, -\frac{12}{5})$; the tangent line there is $3x+4y+15 = 0$. Substituting into $x^2 = 8\alpha y$ and setting the discriminant to zero gives $\alpha = 10/3$, matching List-II's (U) – so $(IV),(U)$ is correct while $(IV),(S) = 21/5$ is the only incorrect pairing.

Answer: (IV), (S) – the only INCORRECT combination