# QuestIQ Academy

JEE Advanced 2021 – Paper 2 – Mathematics – Worked Solutions

**Note:** These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = one or more correct (Q1–6); Section 2 = numerical value, three question-stems (Q7–12); Section 3 = single correct, two paragraphs (Q13–16); Section 4 = non-negative integer (Q17–19).

## Q.1

$S_1 = \{(i,j,k) : i,j,k \in \{1,\ldots,10\}\}$, $S_2 = \{(i,j) : 1 \le i < j+2 \le 10\}$, $S_3 = \{(i,j,k,l) : 1 \le i < j < k < l \le 10\}$, $S_4 = \{(i,j,k,l) : \textit{distinct entries in } \{1,\ldots,10\}\}$. $n_r = |S_r|$.

$n_1 = 10^3 = 1000$ (A true). For $S_2$: $j \le 8$ and $i \le j+1$, giving $\sum_{j=1}^{8}(j+1) = 44$ (B true). $n_3 = \binom{10}{4} = 210 \ne 220$ (C false). $n_4 = 10\cdot 9\cdot 8\cdot 7 = 5040$, $n_4/12 = 420$ (D true).

**Answer: (A), (B), (D)**

## Q.2

$\triangle PQR$ *with sides* $p,q,r$ *opposite* $P,Q,R$.

(A) $\cos P \ge 1 - \frac{p^2}{2qr} \Leftrightarrow q^2+r^2-p^2 \ge 2qr - p^2 \Leftrightarrow (q-r)^2 \ge 0$: always true. (B) Verified via the law of cosines substitution (and confirmed numerically for equilateral, right, and scalene triangles): true. (C) By AM–GM, $\sin Q + \sin R \ge 2\sqrt{\sin Q \sin R}$ always, contradicting the claimed strict reverse inequality: false. (D) For the right triangle $p=3, q=4, r=5$, $\cos Q = (9+25-16)/30 = 0.6 = p/r$ exactly (equality, not strict): false.

**Answer: (A), (B)**

## Q.3

$f : [-\pi/2, \pi/2] \to \mathbb{R}$ *continuous*, $f(0) = 1$, $\int_0^{\pi/3} f(t)\,dt = 0$.

Let $h(x) = \int_0^x f(t)\,dt - \sin 3x$; $h(0) = h(\pi/3) = 0$, so by Rolle's theorem some $c \in (0, \pi/3)$ has $h'(c) = f(c) - 3\cos 3c = 0$ (A true). Similarly $k(x) = \int_0^x f(t)\,dt + \cos 3x - 1 + 6x/\pi$ satisfies $k(0) = k(\pi/3) = 0$, giving $f(c) - 3\sin 3c = -6/\pi$ for some $c$ (B true). Writing $F(x) = \int_0^x f(t)\,dt \approx x$ near 0: $\dfrac{xF(x)}{1-e^{x^2}} \to \dfrac{x^2}{-x^2} = -1$ (C true); $\dfrac{\sin x \cdot F(x)}{x^2} \to \dfrac{x\cdot x}{x^2} = 1 \ne -1$ (D false).

**Answer: (A), (B), (C)**

## Q.4

*$y' + \alpha y = xe^{\beta x}$, $y(1) = 1$; $S = \{y_{\alpha,\beta}\}$. Which functions belong to $S$?*

With $\alpha = 1, \beta = -1$: solving gives $y = \frac{x^2}{2}e^{-x} + Ce^{-x}$ where the $x^2/2$ coefficient is structurally forced positive; applying $y(1) = 1$ gives $C = e - \frac{1}{2}$, matching (A) exactly (true); (B)'s negative $x^2/2$ coefficient can never arise (false). With $\alpha = 1, \beta = 1$: solving gives $y = e^x(x/2 - 1/4) + Ce^{-x}$ (structurally forced), matching (C) with $C = e - e^2/4$ (true); (D)'s reversed sign on the $e^x$ term cannot arise (false).

**Answer: (A), (C)**

## Q.5

$\vec{OA} = (2, 2, 1)$, $\vec{OB} = (1, -2, 2)$, $\vec{OC} = \frac{1}{2}(\vec{OB} - \lambda\vec{OA})$, $\lambda > 0$, $|\vec{OB} \times \vec{OC}| = 9/2$.

Direct computation gives $\vec{OB} \times \vec{OC} = \frac{\lambda}{2}(6, -3, -6)$, so $|\vec{OB} \times \vec{OC}| = 4.5\lambda = 4.5 \Rightarrow \lambda = 1$, giving $\vec{OC} = (-\frac{1}{2}, -2, \frac{1}{2})$. Projection of $\vec{OC}$ on $\vec{OA}$: $\vec{OC} \cdot \vec{OA}/|\vec{OA}| = -4.5/3 = -3/2$ (A true). Area $OAB = \frac{1}{2}|\vec{OA} \times \vec{OB}| = \frac{1}{2}(9) = 9/2$ (B true). Area $ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{1}{2}(9) = 9/2$ (C true). Angle between diagonals $\vec{OA} + \vec{OC}$ and $\vec{OC} - \vec{OA}$: $\cos\theta = 1/\sqrt{5} \neq \cos(\pi/3)$ (D false).

**Answer: (A), (B), (C)**

## Q.6

*$E : y^2 = 8x$, $P = (-2, 4)$; $Q, Q'$ on $E$ with $PQ, PQ'$ tangent to $E$; $F$ = focus.*

Tangent $ty = x + 2t^2$ through $P$: $2t^2 - 4t - 2 = 0 \Rightarrow t = 1 \pm \sqrt{2}$. With $F = (2, 0)$: $\vec{FP} \cdot \vec{FQ} = 0$, confirming $\angle F = 90$ in $\triangle PFQ$ (A true). Since $P = (-2, 4)$ lies exactly on the directrix $x = -2$, the two tangents from $P$ are perpendicular (classical directrix property), giving right angle at $P$ in $\triangle QPQ'$ (B true), and the chord of contact $QQ'$ passes through the focus (D true). $PF = \sqrt{16 + 16} = 4\sqrt{2} \neq 5\sqrt{2}$ (C false).

**Answer: (A), (B), (D)**

## Q.7

*[Stem] $R = \{x \geq 0, y^2 \leq 4 - x\}$; $\mathcal{F}$ = circles in $R$ centered on $x$-axis; $C$ = largest.*

For a circle centered at $(h, 0)$ radius $\rho$ tangent to $y^2 = 4 - x$: substituting gives $x^2 - (2h + 1)x + (h^2 + 4 - \rho^2) = 0$, tangency requires discriminant 0: $\rho^2 = (15 - 4h)/4$. Maximizing $\rho$ subject to $h \geq \rho$ (staying in $x \geq 0$) forces $h = \rho$: $4\rho^2 + 4\rho - 15 = 0 \Rightarrow \rho = 1.5$.

**Answer: 1.5**

## Q.8

*[Same stem] $\alpha$ = $x$-coordinate where $C$ meets the parabola.*

With $h = \rho = 1.5$, the double root is at $x = (2h + 1)/2 = 2$.

**Answer: 2**

## Q.9

*[Stem]* $f_1(x) = \int_0^x \prod_{j=1}^{21}(t-j)^j\,dt$. *Find* $2m_1 + 3n_1 + m_1 n_1$ *(local minima/maxima counts).*

$f_1'(x) = \prod_{j=1}^{21}(x-j)^j$ changes sign at $x = j$ only for odd $j$ (even powers don't flip sign). Tracking the sign pattern across $x = 1, \ldots, 21$: local minima occur at $x = 1, 5, 9, 13, 17, 21$ ($m_1 = 6$) and local maxima at $x = 3, 7, 11, 15, 19$ ($n_1 = 5$). So $2(6) + 3(5) + (6)(5) = 12 + 15 + 30 = 57$.
**Answer: 57**

## Q.10

*[Same stem]* $f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450$. *Find* $6m_2 + 4n_2 + 8m_2n_2$.

With $u = x - 1$: $f_2'(x) = 4900u^{48}(u-6)$. Since $u^{48} \geq 0$ always, sign is governed by $(u-6)$ except at $u = 0$ where it merely touches zero without a genuine sign change. So there is exactly one local minimum at $u = 6$ ($x = 7$), no local maxima: $m_2 = 1, n_2 = 0$. $6(1) + 4(0) + 8(0) = 6$.
**Answer: 6**

## Q.11

*[Stem]* $g_1(x) = 1$, $g_2(x) = |4x - \pi|$, $f(x) = \sin^2 x$ *on* $[\pi/8, 3\pi/8]$; $S_i = \int f g_i\,dx$. *Find* $16S_1/\pi$.

$S_1 = \int_{\pi/8}^{3\pi/8} \sin^2 x\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\pi/8}^{3\pi/8}$; the $\sin 2x$ terms are equal at both endpoints and cancel, giving $S_1 = \frac{1}{2}\cdot\frac{\pi}{4} = \pi/8$. So $16S_1/\pi = 2$.
**Answer: 2**

## Q.12

*[Same stem]* *Find* $48S_2/\pi^2$.

Substituting $x = \pi/4 + u$: $\sin^2 x = \frac{1}{2} + \frac{\sin 2u}{2}$; the odd part vanishes by symmetry over $u \in [-\pi/8, \pi/8]$, leaving $S_2 = 2\int_{-\pi/8}^{\pi/8} |u|\,du \cdot 2 = 4 \cdot \frac{(\pi/8)^2}{2} = \pi^2/32$. So $48S_2/\pi^2 = 48/32 = 1.5$.
**Answer: 1.5**

## Q.13

*[Paragraph]* $a_n = (1/2)^{n-1}$, $S_n = 2 - 2^{1-n}$; $C_n$ *centered* $(S_{n-1}, 0)$ *radius* $a_n$; $M$: *disk radius* $r = 1025/513$. $k$ = *number of* $C_n$ *inside* $M$; $l$ = *max non-intersecting subset.*

$C_n \subset M \Leftrightarrow S_n \leq r$; solving $2 - 2^{1-n} \leq 1025/513$ gives $2^{n-1} \leq 513$, satisfied for $n \leq 10$: $k = 10$. Adjacent circles $C_n, C_{n+1}$ always intersect (distance $a_n$ is between $|a_n - a_{n+1}|$ and $a_n + a_{n+1}$), while $C_n, C_{n+2}$ are always disjoint (distance $\frac{3}{2}a_n > \frac{5}{4}a_n$ = sum of radii). Maximum non-intersecting subset: alternate indices, $l = 5$. Then $3k + 2l = 30 + 10 = 40$.
**Answer: (D)** $3k + 2l = 40$

## Q.14

*[Paragraph] $M$ with $r = (2^{199}-1)\sqrt{2}/2^{198}$; $D_n$ centered $(S_{n-1}, S_{n-1})$ radius $a_n$. Count of $D_n$ inside $M$.*

$D_n \subset M \Leftrightarrow \sqrt{2}\, S_{n-1} + a_n \le r$. Solving this inequality (with $r = 2\sqrt{2} - \sqrt{2}\cdot 2^{-198}$) reduces to $2^{198-n} \ge (4+\sqrt{2})/14 \approx 0.387$, which holds for $n \le 199$ (checking $n = 199$: $2^{-1} = 0.5 \ge 0.387$ true; $n = 200$: $2^{-2} = 0.25 < 0.387$ false).

**Answer: (B) 199**

## Q.15

*[Paragraph] $\psi_1(x) = e^{-x} + x$, $\psi_2(x) = x^2 - 2x - 2e^{-x} + 2$, $f(x) = \int_{-x}^{x}(|t| - t^2)e^{-t^2}dt$, $g(x) = \int_0^{x^2} \sqrt{s}\, e^{-s}ds$.*

Substituting $s = t^2$ in $g$: $g(x) = 2\int_0^x t^2 e^{-t^2}dt$. Since $f(x) = 2\int_0^x (t - t^2)e^{-t^2}dt$, adding: $f(x)+g(x) = 2\int_0^x te^{-t^2}dt = 1-e^{-x^2}$. At $x = \sqrt{\ln 3}$: $f+g = 1-\frac{1}{3} = \frac{2}{3} \ne \frac{1}{3}$ (A false). Testing $x = 2$ in the claimed MVT-type relation for $\psi_1$ gives $\alpha \approx 0.568 \notin (1, 2)$ (B false). Since $\psi_2'(x) = 2(\psi_1(x)-1)$ and $\psi_2(0) = 0$, the Mean Value Theorem on $[0, x]$ directly gives $\psi_2(x) = 2x(\psi_1(\beta) - 1)$ for some $\beta \in (0, x)$ (C true, rigorously). $f'(x) = 2x(1 - x)e^{-x^2}$ is positive on $(0, 1)$ but negative on $(1, 3/2)$: $f$ is not increasing throughout $[0, 3/2]$ (D false).

**Answer: (C)**

## Q.16

*[Same paragraph] Which statement is TRUE?*

$\psi_1'(x) = 1 - e^{-x} > 0$ for $x > 0$, so $\psi_1$ is increasing with $\psi_1(0) = 1$, giving $\psi_1(x) > 1$ (A false). Then $\psi_2'(x) = 2(\psi_1(x) - 1) > 0$, so $\psi_2$ increasing with $\psi_2(0) = 0$ gives $\psi_2(x) > 0$ (B false). Using $g(x) = 2x^3/3 - 2x^5/5 + x^7/7 - \cdots$ (alternating, decreasing terms for $x \in (0, 1/2)$): truncating after the negative term $-2x^5/5$ underestimates $g(x)$, so $g(x) > 2x^3/3 - 2x^5/5$, giving $f(x) = 1 - e^{-x^2} - g(x) < 1 - e^{-x^2} - 2x^3/3 + 2x^5/5$ (C false, opposite direction). Truncating after the positive term $x^7/7$ overestimates: $g(x) \le 2x^3/3 - 2x^5/5 + x^7/7$ (D true).

**Answer: (D)**

## Q.17

*Number chosen from $\{1, \ldots, 2000\}$; $p = P($multiple of $3$ or $7)$. Find $500p$.*

Multiples of 3: 666; of 7: 285; of 21: 95. By inclusion–exclusion: $666 + 285 - 95 = 856$. $p = 856/2000 = 0.428$; $500p = 214$.

**Answer: 214**

## Q.18

*$E : \frac{x^2}{16} + \frac{y^2}{9} = 1$; maximize distance between $M(P, Q)$ and $M(P, Q')$ over distinct $P, Q, Q' \in E$.*

$M(P,Q) - M(P,Q') = \frac{1}{2}(Q - Q')$, independent of $P$. So we maximize $\frac{1}{2}|Q - Q'|$; the largest chord of an ellipse is the major axis, length $2a = 8$, giving maximum distance $8/2 = 4$.
**Answer: 4**

## Q.19

$I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx$. *Find* $9I$.

$h(x) = \sqrt{10x/(x+1)}$ is increasing; $h(x) = k \Rightarrow x = k^2/(10 - k^2)$, giving crossings at $x = 1/9$ ($k = 1$), $x = 2/3$ ($k = 2$), $x = 9$ ($k = 3$), with $h(10) \approx 3.015 < 4$. So $I = 0 \cdot \frac{1}{9} + 1 \cdot (\frac{2}{3} - \frac{1}{9}) + 2 \cdot (9 - \frac{2}{3}) + 3 \cdot 1 = \frac{5}{9} + \frac{50}{3} + 3 = \frac{182}{9}$. $9I = 182$.
**Answer: 182**