QuestIQ Academy

JEE Advanced 2020 – Paper 2 – Mathematics – Worked Solutions

Note: These are QuestIQ Academy's own worked solutions to the Mathematics section only of this paper (Physics and Chemistry are outside QuestIQ Academy's Math-only scope). Section 1 = numerical (single digit 0–9); Section 2 = one or more correct; Section 3 = numerical value.

Q.1

$S = \{z \in \mathbb{C} : z^4 - |z|^4 = 4iz^2\}$. *Find the minimum of* $|z_1 - z_2|^2$ *over* $z_1, z_2 \in S$ *with* $\mathrm{Re}(z_1) > 0$, $\mathrm{Re}(z_2) < 0$.

Let $w = z^2 = u+iv$; since $|z|^2 = |w|$, we get $|z|^4 = |w|^2 = u^2+v^2$. The equation $w^2-|w|^2 = 4iw$ becomes, separating real/imaginary parts,

$$-2v^2 = -4v, \qquad 2uv = 4u$$

giving $v = 0, u = 0$ (i.e. $w = 0$) or $v = 2$ (with u free). So $w = u+2i$ for any real u, i.e. $\mathrm{Im}(z^2) = 2$. Writing $z = x+iy$: $\mathrm{Im}(z^2) = 2xy = 2 \Rightarrow xy = 1$. So $S = \{x+iy : xy = 1\} \cup \{0\}$ (the point 0 has $\mathrm{Re} = 0$, excluded by the strict sign conditions). Take $z_1 = x_1 + \frac{i}{x_1}$ ($x_1 > 0$), $z_2 = -t - \frac{i}{t}$ ($t > 0$). By symmetry the minimum of $(x_1+t)^2 + (1/x_1+1/t)^2$ occurs at $x_1 = t = s$, giving $4s^2 + 4/s^2$, minimized at $s = 1$ (from $8s - 8/s^3 = 0$), value 8.
Answer: 8

Q.2

Missile hits target with probability 0.75*; need at least* 3 *hits to destroy it. Find the minimum number of missiles so that* $P(\textit{destroyed}) \geq 0.95$.

For $X \sim \mathrm{Bin}(n, 0.75)$, need $P(X \geq 3) \geq 0.95$. $n = 5$: $P(X \leq 2) = 0.25^5 + 5(0.75)(0.25)^4 + 10(0.75)^2(0.25)^3 \approx 0.1035 \Rightarrow P(X \geq 3) \approx 0.8965$ — not enough. $n = 6$: $P(X \leq 2) = 0.25^6 + 6(0.75)(0.25)^5 + 15(0.75)^2(0.25)^4 \approx 0.0376 \Rightarrow P(X \geq 3) \approx 0.9624 \geq 0.95$ — works.
Answer: 6

Q.3

Circle $x^2+y^2 = r^2$ $(r > \frac{\sqrt{5}}{2})$, *chord* PQ *on* $2x+4y = 5$. *Circumcentre of* $\triangle OPQ$ *lies on* $x+2y = 4$. *Find* r.

Foot of perpendicular from O to PQ: $M = \left(\dfrac{2\cdot 5}{20}, \dfrac{4\cdot 5}{20}\right) = \left(\frac{1}{2}, 1\right)$, and $|OM| = \frac{\sqrt{5}}{2}$. The perpendicular bisector of PQ (which contains O, M, and the circumcentre C) is the line through the origin perpendicular to PQ: $y = 2x$. Intersecting with $x+2y=4$: $x+4x=4 \Rightarrow x=\frac{4}{5}$, $y=\frac{8}{5}$, so $C=\left(\frac{4}{5},\frac{8}{5}\right)$. $R=|OC|=\sqrt{(4/5)^2+(8/5)^2}=\frac{4\sqrt{5}}{5}$; $|CM|=\left|\frac{4}{5}-\frac{1}{2}\right|\sqrt{5}=\frac{3\sqrt{5}}{10}$ (moving along direction $(1,2)$). Since $CM \perp PQ$: $R^2=|CM|^2+|MP|^2$ where $|MP|^2=r^2-|OM|^2=r^2-\frac{5}{4}$:

$$\left(\frac{3\sqrt{5}}{10}\right)^2+r^2-\frac{5}{4}=\left(\frac{4\sqrt{5}}{5}\right)^2 \Rightarrow 0.45+r^2-1.25=3.2 \Rightarrow r^2=4$$

Answer: $r=2$

Q.4

A is 2×2 with trace$(A)=3$, trace$(A^3)=-18$. Find $\det A$.

Eigenvalues $\lambda_1+\lambda_2=3$, $\det A=\lambda_1\lambda_2=d$. $\lambda_1^3+\lambda_2^3=(\lambda_1+\lambda_2)^3-3\lambda_1\lambda_2(\lambda_1+\lambda_2)=27-9d=-18 \Rightarrow d=5$.

Answer: 5

Q.5

$f(x)=|2x-1|+|2x+1|$ on $(-1,1)$; $g(x)=x-[x]$. For $f\circ g$, let c = number of points of discontinuity, d = number of points of non-differentiability on $(-1,1)$. Find $c+d$.

$g(x)=x+1$ for $x\in(-1,0)$, $g(x)=x$ for $x\in[0,1)$; range of g is $[0,1)$, and g jumps at $x=0$ (left limit 1, value 0). For $y\in[0,1)$: $f(y)=2$ for $y<\frac{1}{2}$, $f(y)=4y$ for $y\geq\frac{1}{2}$ (continuous but with a corner at $y=\frac{1}{2}$, slopes 0 vs 4). $(f\circ g)$ is continuous everywhere except at $x=0$ (since $\lim_{x\to 0^-} f(g(x))=f(1^-)=4\neq f(g(0))=f(0)=2$): $c=1$. Non-differentiable points: $x=0$ (discontinuity), plus wherever $g(x)=\frac{1}{2}$, i.e. $x=\frac{1}{2}$ and $x=-\frac{1}{2}$ (the kink of f transfers there). So $d=3$.

Answer: $c+d=1+3=4$

Q.6

Evaluate $\displaystyle\lim_{x\to\pi/2}\frac{4\sqrt{2}(\sin 3x+\sin x)}{\left(2\sin 2x\sin\frac{3x}{2}+\cos\frac{5x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{3x}{2}\right)}$.

Direct substitution gives $\frac{0}{0}$; the numerator and its first derivative both vanish at $x=\pi/2$, as does the denominator and its first derivative, so the limit is resolved by two applications of L'Hôpital's rule (equivalently, comparing the leading quadratic terms of the Taylor expansions of numerator and denominator about $x=\pi/2$).

Answer: 8

Q.7

$f:\mathbb{R}\to\mathbb{R}$ differentiable, $f(0)=1$, $f'(x)=\dfrac{f(x)}{b^2+x^2}$ $(b\neq 0)$. Which statements are TRUE?

Separating variables: $\ln f(x)=\frac{1}{b}\arctan\frac{x}{b} \Rightarrow f(x)=\exp\left(\frac{1}{b}\arctan\frac{x}{b}\right)$. Since $f(x)>0$ and $b^2+x^2>0$ always, $f'(x)=f(x)/(b^2+x^2)>0$ for *every* x and *every* $b\neq 0$ — so f is always

strictly increasing, regardless of the sign of b. (A) If $b > 0$, f increasing — TRUE. (B) If $b < 0$, f is still increasing (not decreasing) — FALSE. (C) $f(x)f(-x) = \exp\left(\frac{1}{b}\arctan\frac{x}{b} - \frac{1}{b}\arctan\frac{x}{b}\right) = 1$ — TRUE (using arctan is odd). (D) Since f is strictly increasing and non-constant, $f(x) \neq f(-x)$ for $x \neq 0$ — FALSE.
Answer: (A), (C)

Q.8

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $a > 1$, $b < a$. *Tangent at* P *(first quadrant) passes through* $(1, 0)$*; normal at* P *has equal intercepts.* Δ = *area of triangle formed by tangent, normal and* x*-axis;* e = *eccentricity.*

Tangent at $P = (x_0, y_0)$: $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$; through $(1, 0)$: $x_0 = a^2$. On the hyperbola: $y_0^2 = b^2(a^2 - 1)$. Normal intercepts: x-intercept $= x_0\frac{a^2+b^2}{a^2} = a^2 + b^2$ (using $x_0 = a^2$); y-intercept $= y_0\frac{a^2+b^2}{b^2}$. Equal intercepts $\Rightarrow x_0/a^2 = y_0/b^2 \Rightarrow y_0 = b^2$. Combining with $y_0^2 = b^2(a^2 - 1)$: $b^4 = b^2(a^2 - 1) \Rightarrow b^2 = a^2 - 1$. So $e^2 = 1 + b^2/a^2 = 2 - 1/a^2$; as a ranges over $(1, \infty)$, $e^2 \in (1, 2)$, i.e. $1 < e < \sqrt{2}$ — (A) TRUE, (B) FALSE. Triangle vertices: P, $(1, 0)$, and normal's x-intercept $(a^2 + b^2, 0) = (2a^2 - 1, 0)$. Base $= |2a^2 - 1 - 1| = 2(a^2 - 1)$, height $= y_0 = b^2$. $\Delta = \frac{1}{2} \cdot 2(a^2 - 1) \cdot b^2 = (a^2 - 1)b^2 = b^2 \cdot b^2 = b^4$ — (D) TRUE, (C) FALSE.
Answer: (A), (D)

Q.9

$f(x + y) = f(x) + f(y) + f(x)f(y)$, $f(x) = xg(x)$, $\lim_{x\to 0} g(x) = 1$. *Which statements are TRUE?*

Let $h(x) = 1 + f(x)$; then $h(x + y) = h(x)h(y)$. Since $f(x)/x = g(x) \to 1$, h is differentiable at 0 with $h'(0) = 1$, $h(0) = 1$, so $h(x) = e^x$, i.e. $f(x) = e^x - 1$ (the unique solution of Cauchy's exponential equation with this initial slope). f is smooth everywhere: (A) TRUE. $g(x) = (e^x - 1)/x$ extends to an entire analytic (hence everywhere differentiable) function with $g(0) = 1$: (B) TRUE. $f'(x) = e^x$, so $f'(1) = e \neq 1$: (C) FALSE. $f'(0) = e^0 = 1$: (D) TRUE.
Answer: (A), (B), (D)

Q.10

$(1, 0, -1)$ *is the mirror image of... i.e.* $(3, 2, -1)$ *is the mirror image of* $(1, 0, -1)$ *in the plane* $\alpha x + \beta y + \gamma z = \delta$, *with* $\alpha + \gamma = 1$. *Which are TRUE?*

Midpoint $(2, 1, -1)$ lies on the plane: $2\alpha + \beta - \gamma = \delta$. Direction $(2, 2, 0) \parallel (1, 1, 0)$ must be parallel to the normal (α, β, γ): $\alpha = \beta = k$, $\gamma = 0$. With $\alpha + \gamma = 1$: $k = 1$, so $\alpha = 1, \beta = 1, \gamma = 0$, and $\delta = 2(1) + 1 - 0 = 3$. $\alpha + \beta = 2$ (TRUE); $\delta - \gamma = 3$ (TRUE); $\delta + \beta = 4$ (TRUE); $\alpha + \beta + \gamma = 2 \neq \delta = 3$ (FALSE).
Answer: (A), (B), (C)

Q.11

$\overrightarrow{PQ} = a\hat{i} + b\hat{j}$, $\overrightarrow{PS} = a\hat{i} - b\hat{j}$ $(a, b > 0)$, *parallelogram* $PQRS$ *area* = 8. $\vec{u}, \vec{v}$ = *projections of* $\vec{w} = \hat{i} + \hat{j}$ *along* $\overrightarrow{PQ}, \overrightarrow{PS}$. *If* $|\vec{u}| + |\vec{v}| = |\vec{w}|$, *which are TRUE?*

Area $= |a(-b) - ba| = 2ab = 8 \Rightarrow ab = 4$. $|\vec{u}| = \dfrac{a+b}{\sqrt{a^2+b^2}}$, $|\vec{v}| = \dfrac{|a-b|}{\sqrt{a^2+b^2}}$; setting $|\vec{u}|+|\vec{v}| = \sqrt{2}$ and simplifying (splitting on $a \gtrless b$) forces $a = b$ in both cases. With $a = b$ and $ab = 4$: $a = b = 2$. (A) $a + b = 4$ TRUE. (B) $a - b = 0 \neq 2$ FALSE. (C) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS} = (2a, 0) = (4, 0)$, length 4 — TRUE. (D) $\vec{w} = (1, 1)$ is parallel to $\overrightarrow{PQ} = (2, 2)$ itself (not the bisector, which would be along $(1, 0)$) — FALSE.
Answer: (A), (C)

Q.12

With the binomial-sum definitions given for $f(m,n,p)$ and $g(m,n) = \sum_p f(m,n,p)/\binom{n+p}{p}$, which identities hold?

Direct computation gives $g(1,0) = 2$, $g(1,1) = 4$, $g(2,1) = g(1,2) = 8$ — matching the closed form $g(m,n) = 2^{m+n}$ exactly. With $g(m,n) = 2^{m+n}$: (A) $g(m,n) = g(n,m)$: both equal 2^{m+n} — TRUE. (B) $g(m,n+1) = g(m+1,n)$: both equal 2^{m+n+1} — TRUE. (C) $g(2m,2n) = 2\,g(m,n)$: LHS $= 4^{m+n}$, RHS $= 2^{m+n+1}$, equal only when $m+n = 1$ — FALSE in general. (D) $g(2m,2n) = (g(m,n))^2$: LHS $= 4^{m+n} = (2^{m+n})^2 =$ RHS — TRUE.
Answer: (A), (B), (D)

Q.13

Engineer visits a factory exactly 4 days in the first 15 days of June, no two visits on consecutive days. Number of ways.

Choosing k non-consecutive days out of n: $\binom{n-k+1}{k}$. Here $n = 15$, $k = 4$: $\binom{12}{4} = 495$.
Answer: 495

Q.14

6 people into 4 rooms, each room 1 or 2 people. Number of ways.

With 4 rooms and totals 1 or 2 each summing to 6, exactly 2 rooms must have 2 people and 2 rooms 1 person. Choose which 2 rooms are the "double" rooms: $\binom{4}{2} = 6$. Distribute 6 labelled people into rooms of sizes $2, 2, 1, 1$: $\dfrac{6!}{2!\,2!\,1!\,1!} = 180$. Total $= 6 \times 180 = 1080$.
Answer: 1080

Q.15

Two dice rolled repeatedly until the sum is prime or a perfect square. Given the perfect square occurs first, let $p =$ probability that square is odd. Find $14p$.

Terminating sums (out of 36): primes $2, 3, 5, 7, 11$ with weights $1, 2, 4, 6, 2$; squares $4, 9$ with weights $3, 4$. Given the process stops at a square, the relative chance between 4 and 9 depends only on their own weights: $p = \dfrac{P(9)}{P(4) + P(9)} = \dfrac{4}{3+4} = \dfrac{4}{7}$. $14p = 14 \cdot \dfrac{4}{7} = 8$.
Answer: 8

Q.16

$f(x) = \dfrac{4^x}{4^x+2}$ *on* $[0,1]$*. Find* $f(\frac{1}{40}) + f(\frac{2}{40}) + \cdots + f(\frac{39}{40}) - f(\frac{1}{2})$*.*

Key identity: $f(x)+f(1-x)=1$ for all x (direct algebra). Pairing k with $40-k$ for $k=1,\ldots,19$ gives 19 pairs summing to 1 each; the unpaired middle term is $f(\frac{20}{40}) = f(\frac{1}{2})$.

$$\sum_{k=1}^{39} f\left(\tfrac{k}{40}\right) = 19 + f\left(\tfrac{1}{2}\right)$$

Subtracting $f(\frac{1}{2})$ gives 19.
Answer: 19

Q.17

f differentiable, f' continuous, $f(\pi)=-6$, $F(x)=\int_0^x f(t)\,dt$, and $\int_0^\pi \big(f'(x)+F(x)\big)\cos x\,dx = 2$. Find $f(0)$.

Integrating by parts: $\int_0^\pi f'(x)\cos x\,dx = [f(x)\cos x]_0^\pi + \int_0^\pi f(x)\sin x\,dx = \big(-f(\pi)-f(0)\big) + \int_0^\pi f(x)\sin x\,dx = (6-f(0)) + \int_0^\pi f(x)\sin x\,dx$. $\int_0^\pi F(x)\cos x\,dx = [F(x)\sin x]_0^\pi - \int_0^\pi f(x)\sin x\,dx = -\int_0^\pi f(x)\sin x\,dx$. Adding, the $\sin x$ integrals cancel: total $= 6 - f(0) = 2 \Rightarrow f(0) = 4$.
Answer: 4

Q.18

$f(\theta) = (\sin\theta+\cos\theta)^2 + (\sin\theta-\cos\theta)^4$ *on* $(0,\pi)$*. Local minima occur at* $\theta \in \{\lambda_1\pi,\ldots,\lambda_r\pi\}$*. Find* $\sum \lambda_i$*.*

Let $\psi = \theta + \frac{\pi}{4}$. Then $\sin\theta+\cos\theta = \sqrt{2}\sin\psi$ and $(\sin\theta-\cos\theta)^2 = 2\cos^2\psi$, so

$$f = 2\sin^2\psi + 4\cos^4\psi, \qquad \frac{df}{d\psi} = 4\sin\psi\cos\psi(1-4\cos^2\psi)$$

For $\theta\in(0,\pi)$, $\psi \in (\frac{\pi}{4}, \frac{5\pi}{4})$. Critical points: $\cos\psi = \frac{1}{2}$ ($\psi = \frac{\pi}{3}$), $\cos\psi = 0$ ($\psi = \frac{\pi}{2}$), $\cos\psi = -\frac{1}{2}$ ($\psi = \frac{2\pi}{3}$), $\sin\psi = 0$ ($\psi = \pi$) — i.e. $\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$. Sign analysis of $\frac{df}{d\psi}$ across these four points shows f decreasing $\to$ increasing at $\psi = \frac{\pi}{3}$ and at $\psi = \frac{2\pi}{3}$ (local minima), and increasing $\to$ decreasing at $\psi = \frac{\pi}{2}, \pi$ (local maxima). So minima at $\theta = \frac{\pi}{12}$ and $\theta = \frac{5\pi}{12}$: $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$.

$$\lambda_1+\lambda_2 = \frac{1}{12}+\frac{5}{12} = \frac{6}{12} = \frac{1}{2}$$

Answer: $\dfrac{1}{2} = 0.50$